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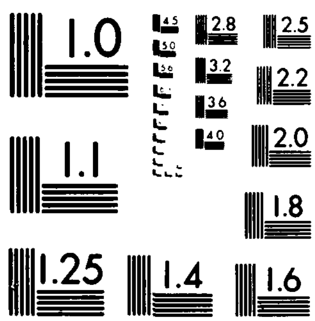
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This technical report has been reviewed and is approved for publication.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A major aim of this paper is the problem of finding function spaces (closed with respect to multiplication and differentiation) to which coefficients $a(x,t)$ of the linear partial differential operator,

$$P(x,t,\partial/\partial x,\partial/\partial y) = (\partial/\partial x)^2 - a(x,t)(\partial/\partial t)$$

(an operator like the one-dimensional heat operator), might belong so that one may find in these spaces only one solution $u(x,t)$ of the equation,

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20. ABSTRACT (Continued)

$$P(x,t,\partial/\partial x,\partial/\partial y)u(x,t) = 0,$$

satisfying the boundary conditions

$$u(0,t) = 0$$

and

$$(\partial/\partial x)u(0,t) = 0.$$

It is interesting that although in the case where V is the space of analytic functions, the Holmgrem uniqueness theorem implies that the only solution of the above boundary value problem is the identically zero solution, Cohen showed that when a(x,t) belongs to a larger vector space (the space of infinitely differentiable functions), one can find in this larger space a nonzero solution.

In this paper nonuniqueness has been obtained for spaces smaller than the space of infinitely differentiable functions, which is an improvement of Cohen's nonuniqueness result. In the course of developing these results we made a study of some of the many function spaces lying between the space of infinitely differentiable functions and the space of real analytic functions. These are generalizations of the spaces studied by Gevrey, Friedman, and Hormander. Because the very definition of these spaces depends on the growth of derivatives, we include for completeness a proof of the formula for the nth derivative of the composition of two functions.

B

PREFACE

The results contained in this report are intended to aid researchers in posing boundary value problems modeling biological or physical phenomena in the appropriate function spaces. If too small a function space is selected, one might not have existence of a solution. Also, as this article points out, if too large a function space is selected, one does not have uniqueness of even the Cauchy Problem, which is the boundary value problem arising, for example, in the initial value problem of wave propagation.

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§1. PRELIMINARIES

§1.1 Introduction

The Holmgren uniqueness theorem (e.g., Hörmander [13], Theorem 5.3.1) gives us a technique for studying the propagation of zero sets of solutions of homogeneous linear partial differential equations across a noncharacteristic hyperplane when the coefficients of the associated linear partial differential operator are analytic.

It is the purpose of this paper to generalize the construction in Theorem 8.9.2 of Hörmander [13] and show the nonpropagation of zero sets of solutions $u(x,t)$ of

$$P(\partial/\partial x)u(x,t) - a(x,t)(\partial/\partial t)u(x,t) = 0$$

across the *noncharacteristic hyperplane* $x = 0$ even when $u(x,t)$ vanishes identically for $x \leq 0$, where $P(\partial/\partial x)$ is an arbitrary polynomial of positive degree in $(\partial/\partial x)$ and the coefficient $a(x,t)$ and the function $u(x,t)$ belong to a certain Frechet space of infinitely differentiable functions containing the real analytic functions and contained properly in the space $C^\infty(R_x x R_t)$. More precisely, the coefficients will be in the space $\gamma^{(\overline{\delta},\overline{\eta})}(R_x x R_t)$, where we define the space $\gamma^{(\overline{\delta},\overline{\eta})}(\Omega)$ for every open subset Ω of R^n, n-dimensional space, by the following definition for all n-tuples $\overline{\delta}$ and $\overline{\eta}$ of positive numbers.

Definition 1.1.1 *We say that a function* f *in* $C^\infty(\Omega)$ *is in* $\gamma^{(\overline{\delta},\overline{\eta})}(\Omega)$, *where* $\overline{\delta}$ *and* $\overline{\eta}$ *are* n-*tuples of positive numbers*, *provided that for every*

compact subset K of Ω and every $\varepsilon > 0$ the seminorms of f defined by the rule,

$$\|f\|_{(K,\varepsilon)}^{(\delta,\bar{n})} = \sup\{|D^{\alpha}f(x)|\left[\prod_{j=1}^{n} |\alpha_j|^{-\delta_j|\alpha_j|^{n_j}}\right]\varepsilon^{-|\alpha|} :$$

$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n = \|\alpha\|$,

$x \in K$, $\alpha \in N^n$, the set of

n-tuples of nonegative integers$\}$ (1.1.1),

are finite.

If $\bar{n}$ is an n-tuple, each entry of which is 1, then we write $\gamma^{(\delta)}(\Omega)$ instead of $\gamma^{(\delta,\bar{n})}(\Omega)$.

In case there is no chance of confusion we write $\|f\|_{(K,\varepsilon)}$ instead of $\|f\|_{(K,\varepsilon)}^{(\delta,n)}$.

In section 1.2 we give statements and proofs (for the sake of completeness) of two formulae for the nth derivative of the composition of two functions. Jensen's inequality is used to obtain a more precise version of the Faa di Bruno formula.

In section 1.3 we introduce the spaces $\gamma^M(\Omega)$, for every mapping M from N^n into R_+^n, the set of x in R^n such that $x_i > 0$ for $i = 1, 2, \ldots, n$, for all open subsets Ω of R^n. We say that an $f(x)$ in $C^\infty(\Omega)$ is in $\gamma^M(\Omega)$ if for every compact subset K of Ω and every $\varepsilon > 0$ there is a $C > 0$ such that $M(\alpha) = (M_1(\alpha), M_2(\alpha), \ldots, M_n(\alpha))$ implies

$$|D^{\alpha}f(x)|\varepsilon^{-\|\alpha\|}\prod_{k=1}^{n}M_k(\alpha)^{-1} \leq C \qquad (1.1.2)$$

for all x in K and all α in N^n. We determine the composition of a function in $\gamma^{M_2}(\Omega,R)$ and a function in $\gamma^{M_2}(R,R)$ when the Ω is an open subset of R^n and

$$x \to (M_k(x)^{1/x})/x \qquad (1.1.3)$$

is an increasing function of $x \geq 1$ for $k = 1, 2$. In particular this enables us to determine a function space containing the composition of two real valued functions in $\gamma^{(\delta)}(R)$.

In section 2.1 we use Paley-Wiener theorems to determine the Fourier Transforms of functions in $\gamma^M(R^n)$ with support in a closed ball.

In section 2.2 we give techniques for providing that a function belongs to $\gamma^M(\Omega)$.

In section 2.3 we introduce the space Γ^M and study the natural locally convex topologies on $\gamma^M(\Omega)$ and $\Gamma^M(\Omega)$. We observe that $\gamma^M(\Omega)$ and $\Gamma^M(\Omega)$ are both Frechet spaces with $\gamma^M(\Omega) \subset \Gamma^M(\Omega)$.

In section 3.1 we prove that the construction of our generalization of Theorem 8.9.2 of Hörmander [13] cannot produce functions $a(x,t)$ and $u(x,t)$ in $\gamma^{(\delta)}(R_x \times R_t)$ such that

$$P(\partial/\partial x)u(x,t) - a(x,t)(\partial/\partial t)u(x,t) = 0 \qquad (1.1.4)$$

$$u(x,t) = 0 \qquad \text{for } x \leq 0 \qquad (1.1.5)$$

and every point of $x = 0$ is in support of $u(x,t)$.

In section 3.2 we produce for every polynomial $P(\partial/\partial x)$ in $\partial/\partial x$ functions $a(x,t)$ and $u(x,t)$ in $\gamma^{(\overline{\delta},\overline{n})}(R_x \times R_t)$ such that (1.1.4) and (1.1.5) are satisfied and yet every point of $x = 0$ is in the support of $u(x,t)$.

The properties of the space $\gamma^{(\overline{\delta})}(\Omega) = \gamma^{(\overline{\delta},\overline{1})}(\Omega)$, where $\overline{1} = (1, 1, \ldots, 1)$ are studied in reference 4. Some of these are stated without proof in the introduction to reference 3.

A trivial consequence of our main results contained in sections 3.1 and 3.2 is the nonextendability of Rado's Theorem to $\gamma^{(\delta)}(R^n)$. In other words, we give an example of a function $\psi(x)$ in $C^\infty(R^n)$ which is in $\gamma^{(\delta)}(R^n - Z)$, where Z is the zero set of ψ but which is not in $\gamma^{(\delta')}(R^n)$ for any δ'.

The main result, however, is the nonextendability of the Holmgren uniqueness theorem to operators whose coefficients are in $\gamma^{(\overline{\delta},\overline{n})}(\Omega)$.

§1.2 Formulae for the nth Derivative of Compositions of Functions

There are two formulae for the nth derivative of a composition of two functions in the literature. Both appear in the table of Gradshteyn and Ryzhik [10]. The Jensen-Vollers formula is found in the table of Adams and Hippisley [1]. Jensen's result [14] and Vollers' result [24] are the same, but were evidently discovered independently. Some pre-Jensen contributors to the theory have published findings [6-9, 21]. The post-Jensen, pre-Vollers contributors have also published [11, 12, 22]. The other formula, a variation of which we prove in this section and use in further developments has an older history. A paper giving applications of this formula was written by Teixeira [23] in 1885 but was discovered by Faa di Bruno [2] in 1857. Königsberger [15] in 1886 wrote a paper giving the applications to functions of several variables. Many people since then have corrected and clarified the old formulae and have given elegant proofs of their correctness. Among them are Dresden [5, 1943], Riordan [18, 1943], McKiernan [16, 1956], and Pandres [17, 1957]. The author gives a distribution theory proof of the Jensen-Vollers formula and states and proves using induction a slightly different version of the Faa di Bruno formula which seems to be useful in the calculations.

Theorem 1.2.1 *Let* $\phi(x)$ *be a function which is* C^∞ *in an open subset of* R. *Let* F *be a function which is in* $C^\infty(\phi(\Omega))$, *the space of functions which are* C^∞ *in some open set containing* $\phi(\Omega)$, *in case* ϕ *is real valued, or is in* $H(\phi(\Omega))$, *the space of functions holomorphic in some open set in the complex plane containing* $\phi(\Omega)$, *in case* ϕ *is complex valued*.

Then $f(x) = F(\phi(x))$ implies

$$(\frac{d}{dx})^n f(x) = \sum_{k=1}^{n} \frac{U_{(n,k)}}{k!} F^{(k)}(y) \tag{1.2.1}$$

<u>where</u> $y = \phi(x)$ <u>and</u>

$$U_{(n,k)} = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} y^{k-j} (\frac{d}{dx})^n (y^j). \tag{1.2.2}$$

<u>Proof</u>. Let $P(n)$ denote the statement (1.2.1). That $P(1)$ holds is obvious. We show that $P(n)$ implies $P(n+1)$.

Assuming $P(n)$ we deduce that

$$(\frac{d}{dx})^{n+1} f(x) = (\frac{d}{dx})(U_{(n,1)}) F^{(1)}(y) +$$

$$\sum_{k=2}^{n} \frac{\frac{d}{dx}(U_{(n,k)}) + k\, U_{(n,k-1)} \frac{dy}{dx}}{k!} F^{(k)}(y) +$$

$$\frac{U_{(n,n)}}{n!} (\frac{dy}{dx}) F^{(n+1)}(y) \tag{1.2.3}$$

Thus, (1.2.2) implies that what we must show is that

$$U_{(n+1,1)} = \frac{d}{dx}(U_{(n,1)}) \tag{1.2.4}$$

$$U_{(n+1,k)} = \frac{d}{dx}(U_{(n,k)}) + k\, U_{(n,k-1)} \frac{dy}{dx} \tag{1.2.5}$$

for $k = 2, 3, \ldots, n$, and

$$U_{(n+1,n+1)} = (n+1)\, U_{(n,n)} \frac{dy}{dx} \tag{1.2.6}$$

Proof of (1.2.4). By definition $U_{(n,1)} = (d/dx)^n y$. Thus, (1.2.4) is true, since $(d/dx)U_{(n,1)} = (d/dx)^{n+1}y$.

Proof of (1.2.5). The product rule tells us immediately that

$$\frac{d}{dx}\left(U_{(n,k)}\right) = U_{(n+1,k)} +$$

$$\sum_{j=1}^{k} (-1)^{k-j}\binom{k}{j}(k-j)\, y^{k-j-1}\left(\frac{dy}{dx}\right)\left(\frac{d}{dx}\right)^{n}(y^{j}) \tag{1.2.7}$$

We observe that the second term of the right side of (1.2.7) is simply

$$(-1)k\left(\frac{dy}{dx}\right)\sum_{j=1}^{k-1}(-1)^{k-1-j}\binom{k}{j}(k-j)y^{k-1-j}\left(\frac{d}{dx}\right)^{n}(y^{j}) =$$

$$(-1)k\left(\frac{dy}{dx}\right)U_{(n,k-1)}\ , \tag{1.2.8}$$

which proves the validity of (1.2.5).

Proof of (1.2.6). We must show that

$$U_{(n,n)} = n!\ \left(\frac{dy}{dx}\right)^{n} \tag{1.2.9}$$

where $U_{(n,n)}$ is given by (1.2.2).

To prove (1.2.9) we need the following Lemma:

Lemma 1.2.1. For every positive integer n and for all integers $q \in \{0, 1, \ldots, n\}$ we have for every C^{∞} function y of x the relation,

$$0 = \sum_{j=0}^{n+1} (-1)^{n+1-j}\binom{n+1}{j}y^{n+1-j}\left(\frac{d}{dx}\right)^{q}(y^{j})\ . \tag{1.2.10}$$

Proof of Lemma 1.2.1. We proceed by induction on n and note that (1.2.10) is trivial for $q = 0$. Let $P(n,q)$ denote the statement (1.2.10), and observe that $P(1,0)$ and $P(1,1)$ are true. Assume $n > 1$, $q > 0$ and

that $P(m,p)$ is true for all integers $m \in \{1, \ldots, n-1\}$ and all $p \in \{1, \ldots, m\}$ and that $P(n,p)$ is true for all $p \in \{0, \ldots, q-1\}$. Let ψ be an arbitrary test function. Then an integration by parts tells us that

$$\int \psi \left[\sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} y^{n+1-j} \left(\frac{d}{dx}\right)^{q} (y^{j}) \right] dx =$$

$$\int \left(-\frac{d\psi}{dx}\right) \left[\sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} y^{n+1-j} \left(\frac{d}{dx}\right)^{q-1} (y^{j}) \right] dx +$$

$$(n+1) \int \psi \left[\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} y^{n-j} \left(\frac{d}{dx}\right)^{q-1} (y^{j}) \right] \left(\frac{dy}{dx}\right) dx \qquad (1.2.11)$$

By the inductive hypothesis the terms in square brackets in the integrands of the right side of (1.2.11) vanish. Hence, since

$$\int \psi \left[\sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} y^{n+1-j} \left(\frac{d}{dx}\right)^{q} (y^{j}) \right] dx = 0 \qquad (1.2.12)$$

for all test functions ψ, it follows that $P(n,q)$ is true.

Completion of the Proof of (1.2.6). Let ψ be an arbitrary test function. We want to prove that

$$U_{(n+1,n+1)} = (n+1)\left(\frac{dy}{dx}\right) U_{(n,n)} \qquad (1.2.13)$$

for every positive integer n, which will prove (1.2.9). An integration by parts tells us that for all positive integers n

$$\int \psi\, U_{(n+1,n+1)}dx =$$

$$\int(-\psi')\left[\sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}y^{n+1-j}\left(\frac{d}{dx}\right)^n(y^j)\right]dx\ +$$

$$\int \psi\left[(n+1)\ U_{(n,n)}\frac{dy}{dx}\right]dx \tag{1.2.14}$$

But Lemma 1.2.1 applied to (1.2.14) implies that

$$\int \psi\, U_{(n+1,n+1)}dx = \int \psi\left[(n+1)\ U_{(n,n)}\ \frac{dy}{dx}\right]dx \tag{1.2.15}$$

for all test functions ψ. Hence, since the test functions are dense in the dual of the C^∞ functions it follows that (1.2.13) holds. This completes the proof of the theorem.

<u>Corollary</u> 1.2.1 <u>Let</u> $y = \phi(x)$ <u>be a</u> C^∞ <u>function and let</u> p <u>be a positive integer</u>.

<u>Then</u>

$$\left(\frac{d}{dx}\right)^n(y^p) = \sum_{k=1}^{n}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}y^{p-j}\frac{p^{(k)}}{k!}\left(\frac{d}{dx}\right)^n(y^j)$$

$$\text{for } n < p$$

$$\left(\frac{d}{dx}\right)^n(y^p) = \sum_{k=1}^{p}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}y^{p-j}\frac{p^{(k)}}{k!}\left(\frac{d}{dx}\right)^n(y^j)$$

$$\text{for } n \geq p \tag{1.2.16}$$

<u>for all positive integers</u> n, where $p^{(k)} = p(p-1)\ldots(p-(k-1))$.

Proof of Corollary 1.2.1. Apply Theorem 1.2.1 with $F(y) = y^p$.

Theorem 1.2.2 Let $\phi(x)$ be a function which is C^∞ in an open subset U of R. Let F be a function which is in $C^\infty(\phi(U))$, the space of functions which are C^∞ in some open set containing $\phi(U)$, in case ϕ is real valued, or is in $H(\phi(U))$, the space of functions Holomorphic in some open set containing $\phi(U)$ in case ϕ is complex valued. Define $f(x) = F(\phi(x)) = F(y)$, where $y = \phi(x)$, for all x in U. Then

$$\left(\frac{d}{dx}\right)^n f(x) =$$

$$\sum_{m=1}^{n} \sum_{p=1}^{n} \sum_{(i_1,i_2,\ldots i_p)\varepsilon S(n,m,p)} \frac{n!}{i_1! \ldots i_p!} \left[\left(\frac{d}{dy}\right)^m F(y)\right] \prod_{j=1}^{p} \left(\frac{y^{(j)}}{j!}\right)^{i_j} \tag{1.2.17}$$

where

$$S(n,m,p) = \{(i_1,i_2,\ldots,i_p) \varepsilon N \times N \times \ldots \times N:$$

$$i_p \neq 0 \ , \sum_{j=1}^{p} j i_j = n \ \text{and} \ m = \sum_{j=1}^{p} i_j\} \tag{1.2.18}$$

and where N denotes the set of nonnegative integers.

Proof of Theorem 1.2.2. For convenience let i denote $(i_1,\ldots,i_p)$ and let $i!$ denote $i_1!i_2!\ldots i_p!$, where is it stated that i belongs to the set, $S(n,m,p)$, defined by (1.2.18). Then differentiating both sides of (1.2.17) we deduce that

$$\left(\frac{d}{dx}\right)^{n+1} f(x) =$$

$$\sum_{m=1}^{n} \sum_{p=1}^{n} \sum_{i\varepsilon S(n,m,p)} \left[\frac{n!}{i!}\left[\prod_{j=1}^{p}\left(\frac{y(j)}{j!}\right)^{i_j}\right]\right]\left(\frac{dy}{dx}\right) F^{(m+1)}(y) +$$

$$\sum_{m=1}^{n} \sum_{p=1}^{n} \sum_{i\varepsilon S(n,m,p)} \left[\frac{n!}{i!}\frac{d}{dx}\left[\prod_{j=1}^{p}\left(\frac{y(j)}{j!}\right)^{i_j}\right]\right] F^{(m)}(y) \qquad (1.2.19)$$

Let

$$\tilde{i} = (i_1+1,\ i_2,\ \ldots,\ i_p) \qquad (1.2.20)$$

whenever it is stated that i belongs to $S(n,m,p)$.

Collecting terms in (1.2.19) we deduce that

$$\left(\frac{d}{dx}\right)^{n+1} f(x) = \sum_{p=1}^{n} \sum_{i\varepsilon S(n,m,p)} \frac{n!}{i!}\left[\prod_{j=1}^{p}\left(\frac{y(j)}{j!}\right)^{\tilde{i}_j}\right] F^{(n+1)}(y) +$$

$$\sum_{m=2}^{n} \sum_{p=1}^{n} \sum_{i\varepsilon S(n,m-1,p)} \frac{n!}{i!}\left[\prod_{j=1}^{p}\left(\frac{y(j)}{j!}\right)^{\tilde{i}_j}\right] + \sum_{i\varepsilon S(n,m,p)} \binom{n}{i} \frac{d}{dx}\left[\prod_{j=1}^{p}\left(\frac{y(j)}{j!}\right)^{i_j}\right] F^{(m)}(y)$$

$$+ \sum_{p=1}^{n} \sum_{i\varepsilon S(n,1,p)} \frac{n!}{i!}\left(\frac{d}{dx}\right)\left[\prod_{j=1}^{p}\left(\frac{y(j)}{j!}\right)^{i_j}\right] F^{(1)}(y)\ . \qquad (1.2.21)$$

To complete the verification of (1.2.17) by induction, we have to show that

$$\sum_{p=1}^{m} \sum_{i\in S(n+1,n+1,p)} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j}\right] =$$

$$\sum_{p=1}^{m} \sum_{i\in S(n,m,p)} \frac{n!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j}\right] \tag{1.2.22}$$

$$\sum_{p=1}^{n+1} \sum_{i\in S(n+1,m,p)} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j}\right] =$$

$$\sum_{p=1}^{n} \sum_{i\in S(n,m-1,p)} \frac{n!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j}\right] + \sum_{i\in S(n,m,p)} \frac{n!}{i!} \frac{d}{dx} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j}\right] \tag{1.2.23}$$

for $m = 2, \ldots, n$ and

$$\sum_{p=1}^{n+1} \sum_{i\in S(n+1,1,p)} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j}\right] =$$

$$\sum_{p=1}^{n} \sum_{i\in S(n,1,p)} \frac{n!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j}\right] \tag{1.2.24}$$

Proof of (1.2.22). Here we need to know

$S(n+1,n+1,p) = \{(i_1, \ldots, i_p) \in N^p\colon\ i_p \neq 0,\ i_1 + \ldots + i_p = n+1, \text{ and } i_1 + 2i_2 + \ldots + pi_p = n+1\}$. It is obvious that $S(n+1,n+1,p) \neq 0$ implies $i_j = 0$ for $j = 2, \ldots, p$. But $i_p \neq 0$. Hence, $S(n+1,n+1,p) \neq 0$ only if $p = 1$. Thus, $S(n+1,n+1,1) = \{n+1\}$. Since by convention $\sum_{i\in 0} \psi(i) = 0$, we deduce that

$$\sum_{p=1}^{n+1} \sum_{i \in S(n+1,n+1,p)} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j} \right] = \left(\frac{dy}{dx}\right)^{n+1} \tag{1.2.25}$$

The completes the proof of (1.2.22).

Proof of (1.2.24). We need to know

$$S(n,1,p) = \{(i_1, \ldots, i_p) : i_p \neq 0,$$
$$i_1 + \ldots + i_p = 1, \text{ and } i_1 + 2i_2 + \ldots + pi_p = n\} \tag{1.2.26}$$

Since $i_p \neq 0$ and $i_1 + \ldots + i_p = 1$ it is clear that $i_1 = i_2 = \ldots = i_{p-1} = 0$ and $i_p = 1$. It is also clear that $i_1 + 2i_2 + \ldots + pi_p = p = n$ if and only if $p = n$. Thus, $S(n,1,p) \neq \phi$ if and only if $p = n$ and that $S(n,1,n) = \{(0, \ldots, 0, 1)\}$. From this we conclude that

$$\sum_{p=1}^{n+1} \sum_{i \in S(n+1,1,p)} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j} \right] =$$

$$\sum_{i \in S(n+1,1,n+1)} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{n+1} \left(\frac{y(j)}{j!}\right)^{i_j} \right] =$$

$$(n+1)! \left[\frac{y(n+1)}{(n+1)!} \right]^{i_{n+1}} = \left(\frac{d}{dx}\right)^{n+1} y(x),$$

which completes the proof of (1.2.24).

To complete the proof (1.2.23) and, consequently, the theorem, we need some easy lemmas.

Lemma 1.2.2. *For each* i *in the set* $S(n,m,p)$ *with* $i_k \neq 0$ *define* $i^{(k)}$ *in* $S(n+1,m,p)$ *by the rule*

$$i^{(k)} = (i_1, \ldots, i_{k-1}, i_k-1, i_{k+1}+1, i_{k+2}, \ldots, i_p) \tag{1.2.27}$$

for $k = 1, 2, \ldots, p-1$. *For every* i *in* $S(n,m,p)$ *define* $i^{(p)}$ *in* $S(n+1,m,p+1)$ *by the rule*,

$$i^{(p)} = (i_1, \ldots, i_p, 1) \tag{1.2.28}$$

Then for each $i \in S(n,m,p)$

$$\frac{d}{dx}\left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j}\right] =$$

$$\sum_{k=1}^{p-1} (k+1)i_k\left[\prod_{j=1}^{p} \left[\frac{y(j)}{j!}\right]^{i_j^{(k)}}\right] + (p+1)i_p\left[\prod_{j=1}^{p+1} \left[\frac{y(j)}{j!}\right]^{i_j^{(p)}}\right] \tag{1.2.29}$$

for $p = 1, 2, \ldots, n$ and

Proof of Lemma 1.2.2. This is an immediate consequence of the definition of the sets $S(n,m,p)$ and the logarithmic differentiation rule. When $p = 1$, the first sum on the right side of (1.2.29) is automatically zero and

$$\frac{d}{dx}\left[\prod_{j=1}^{1} \left(\frac{y(j)}{j!}\right)^{i_j}\right] = 2i_1 \frac{(y(1))^{i_1-1}}{(1!)^{i_1-1}} \frac{y(2)}{2!},$$

which is exactly the second term on the right side of (1.2.29).

Lemma 1.2.3. Let $\tilde{i}$ be as defined by (1.2.20). Then

$$\sum_{p=1}^{n} \sum_{i\epsilon S(n,m-1,p)} \frac{n!}{\tilde{i}!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{\tilde{i}_j} \right] =$$

$$\sum_{p=1}^{n} \sum_{i\epsilon S(n+1,m,p)} \frac{i_1}{n+1} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j} \right] \qquad (1.2.30)$$

Proof of Lemma 1.2.3. Direct calculation.

Lemma 1.2.4. Let $i^{(p)}$ be defined as in Lemma 1.2.2. Then

$$\sum_{p=1}^{n} \sum_{i\epsilon S(n,m,p)} \frac{n!}{i!} (p+1)i_p \left[\prod_{j=1}^{p+1} \left(\frac{y(j)}{j!}\right)^{i_j^{(p)}} \right] =$$

$$\sum_{p=2}^{n} \sum_{i\epsilon S(n+1,m,p)} \frac{pi_p}{(n+1)} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j^{(p)}} \right] +$$

$$\sum_{p=n+1}^{n+1} \sum_{i\epsilon S(n+1,m,n+1)} \frac{pi_p}{(n+1)} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j^{(p)}} \right] \qquad (1.2.31)$$

Proof. This is just the observation that $i_{p+1}^{(p)} = 1$ for all σ.

Lemma 1.2.5. Let $i^{(k)}$ be as defined in Lemma 1.2.2. Then

$$\sum_{p=2}^{n} \sum_{i\in S(n,m,p)} \frac{n!}{i!} \left[\sum_{k=1}^{p-1} (k+1) i_k \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j^{(k)}} \right] \right] =$$

$$\sum_{p=2}^{n} \sum_{i\in S(n+1,m,p)} \sum_{k=2}^{p-1} \frac{k i_k}{n+1} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j} \right] +$$

$$\sum_{p=2}^{n} \sum_{\substack{i\in S(n+1,m,p) \\ i_p > 1}} \frac{p i_p}{n+1} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j} \right]$$

Proof of Lemma 1.2.5. This is just the observation that $i_p \neq 0$ implies $i_p^{(p-1)} > 1$.

Proof of (1.2.23). Putting together the lemmas we deduce that the right side of (1.2.23) is given by

$$\sum_{p=1}^{n+1} \sum_{i\in S(n+1,m,p)} \sum_{k=1}^{n+1} \frac{k i_k}{n+1} \frac{(n+1)!}{i!} \left[\prod_{j=1}^{p} \left(\frac{y(j)}{j!}\right)^{i_j} \right] \qquad (1.2.32)$$

which is clearly equal to the left side of (1.2.23) since

$$\sum_{k=1}^{n+1} \frac{k i_k}{n+1} = 1 .$$

This completes the proof of the Faa di Bruno formula.

An application of Jensen's inequality produces the following simplification of the Faa di Bruno formula. If F and ϕ satisfy the hypothesis of Theorem 1.2.2, then

$$(\frac{d}{dx})^n f(x) =$$

$$\sum_{m=1}^{n} \sum_{p=[n/m]}^{n} \sum_{i \varepsilon S(n,m,p)} (\frac{n!}{i!}) \left[\prod_{j=1}^{p} (\frac{\phi^{(j)}(x)}{j!})^{i_j} \right] F^{(m)}(y) \quad , \qquad (1.2.33)$$

where $[n/m]$ is the greatest integer not exceeding n/m.

We let $X = \{1, 2, \ldots, p\}$ and $\mu(\{j\}) = i_j/m$. Then $\mu(X) = 1$ implies that

$$\exp(\frac{i_1 + 2i_2 + \ldots + pi_p}{m}) \leq$$

$$\exp(1)(i_1/m) + \exp(2)(i_2/m) + \ldots + \exp(p)(i_p/m) \leq \exp(p) \qquad (1.2.34)$$

Thus, since the left side of (1.2.34) is equal to $\exp(n/m)$ we deduce that

$$n/m \leq p \qquad (1.2.35)$$

if $(i_1, \ldots, i_p)$ is a p-tuple of integers in $S(n,m,p)$.

§1.3. Determination of the Space Containing Compositions of Functions in the Spaces $\gamma^{M_1}(\Omega,R)$ and $\gamma^{M_2}(R)$.

The main result of this section is to show that if M_1 and M_2 are mappings from R_+ into R_+ which satisfy the condition that the mappings

$$x \to M_k(x)^{1/x}/x$$

for $k = 1, 2$ are increasing functions of $x \geq 1$, then $g(x)$ is in $\gamma^{M_1}(\Omega,R)$ and $F(y)$ is in $\gamma^{M_2}(R)$ implies

$$f(x) = F(g(x))$$

is a member of $\gamma^{M_1 M_2}(\Omega)$. We give applications to interesting special cases.

Definition 1.3.1. Let $M: N^n \to R_+^n$ denote a mapping from N^n, the set of n-tuples of nonnegative integers into the set R_+^n, of n-tuples of positive numbers. Let Ω be an open subset of R^n, n dimensional space. Let $\gamma^M(\Omega)$ denote the set of all functions f in $C^\infty(\Omega)$ such that for every compact subset K of Ω and every $\varepsilon > 0$ there is a $C > 0$ such that

$$|D^\alpha f(x)|\,\varepsilon^{-|\alpha|}\left[\prod_{k=1}^{n} M_k(\alpha)\right]^{-1} \leq C$$

for every α in N^n and every $x \in K$

We let

$$\|f\|(K,\varepsilon,M) =$$

$$\sup\left\{ \left|D^{\alpha} f(x)\right| \varepsilon^{-|\alpha|} \prod_{k=1}^{n} M_k(\alpha)^{-1} : x \in K,\ a \in N^n \right\}$$

Remark 1.3.1. *If*

$$M_k(\alpha) = \alpha_k^{\delta_k \alpha_k^{\eta_k}}$$

for $k = 1, 2, \ldots, n$, *then*

$$\gamma^M(\Omega) = \gamma^{(\bar{\delta},\bar{\eta})}(\Omega) \quad ,$$

where $\bar{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$ *and* $\bar{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$.

Remark 1.3.2. *If*

$$M_k(\alpha) = \alpha_k^{\delta_k \alpha_k}$$

for $k = 1, 2, \ldots, n$, *then*

$\gamma^M(\Omega) = \gamma^{\bar{\delta}}(\Omega)$, *where* $\bar{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$.

We begin by providing some useful special cases of our general results.

Theorem 1.3.1. *Let* F *be a member of* H(Range (g)), *where* g *is an arbitrary member of* $\gamma^{(\delta)}(\Omega)$. *Then* $f = F(g)$ *is a member of* $\gamma^{(\delta+1)}(\Omega)$.

Proof. The n^{th} derivative of f is given by (1.2.17). For every compact subset K of Ω, the set $K' = g(K)$ is a compact subset of C, and there exist positive constants A and B such that

$$\left(\frac{d}{dy}\right)^m F(y) \leq A\ B^m\ m! \tag{1.3.1}$$

for all y in K'. Let $\varepsilon > 0$ be given. First suppose Ω is a subset of R^1. Then $y = f(x)$ implies that for every $\varepsilon > 0$ there is a $C > 0$ such that

$$\left|y^{(j)}\right| \leq C\varepsilon^j j^{j\delta} . \tag{1.3.2}$$

Thus, from (1.2.17) we deduce that

$$\left(\frac{d}{dx}\right)^n f(x) \leq$$

$$\sum_{m=1}^{n} \sum_{p=1}^{n} \sum_{i\varepsilon S(n,m,p)} \left(\frac{n!}{i!}\right) (A\ B^m\ m!)\left(\prod_{j=1}^{p} \left(\left(\frac{\varepsilon D}{j}\right)^j (j^{j\delta})\right)^{i_j}\right) C^m, \tag{1.3.3}$$

where D is a positive constant such that

$$\left(\frac{1}{j!}\right) \leq \left(\frac{D}{j}\right)^j . \tag{1.3.4}$$

From (1.3.3) we deduce that

$$\left(\frac{d}{dx}\right)^n f(x) \le$$

$$\sum_{m=1}^{n} \sum_{p=1} \sum_{i\varepsilon S(n,m,p)} \left(\frac{n!}{i!}\right) A\ B\ m!\ (\varepsilon D)^n\ C^m \prod_{j=1}^{p} \left(j^{j(\delta-1)i_j}\right) \tag{1.3.5}$$

for all x in K.

Lemma 1.3.1. For all positive integers p and all $\delta > 0$ we have

$$\left[\prod_{j=1}^{p} j^{(j\delta-j)i_j}\right] \le p^{n\delta-n} \tag{1.3.6}$$

Proof of Lemma 1.3.1. This is a trivial consequence of the inequality of Jensen, which states that if μ is a Borel measure on a σ-algebra on X such that $\mu(X) = 1$, f is a bounded μ-measurable function on X and ϕ is convex on $f(X)$, then

$$\phi\left(\int_X f d\mu\right) \le \int_X \phi(f) d\mu \tag{1.3.7}$$

where we take $X = \{1, 2, \ldots, p\}$, $\mu(\{j\}) = (j\delta-j)i_j/(n\delta-n)$, $f(j) = \ell n(j)$, and $\phi(s) = \exp((n\delta-n)s)$. Then

$$\int_X f d\mu = \sum_{j=1}^{p} \frac{(j\delta-j)i_j\ \ell n(j)}{(n\delta-n)}$$

Applying (1.3.7) to the integral defined above we deduce that the left side of (1.3.6) is dominated by

$$\sum_{j=1}^{p} \left[\frac{(j\delta-j)i_j}{n\delta-n}\right] j^{n\delta-n} \le \sum_{j=1}^{p} \left(\frac{(j\delta-j)i_j}{n\delta-n}\right) j^{n\delta-n},$$

which is easily seen to be bounded above by the right side of (1.3.6).

Combining (1.3.5) and the result of Lemma 1.3.1 we deduce that

$$\left|\left(\frac{d}{dx}\right)^n f(x)\right| \le$$

$$\sum_{m=1}^{n} \sum_{p=1}^{n} \sum_{i\varepsilon S(n,m,p)} \left(\frac{n!}{i!}\right) A\, B^m m! (\varepsilon D)^n\, C^m\, p^{n\delta-n} \tag{1.3.8}$$

for all x in K.

But it is easy to see that

$$\sum_{i\varepsilon S(n,m,p)} \left(\frac{n!}{i!}\right) \le \frac{n!\ p^m}{m!} \tag{1.3.9}$$

combining (1.3.8) and (1.3.9) and applying Stirling's inequality, we observe that for every $\varepsilon > 0$ there is a $C_1 > 0$ such that for all x in K

$$\left|\left(\frac{d}{dx}\right)^n f(x)\right| \le C_1 \varepsilon^n\, n^{n(\delta+1)} \tag{1.3.10}$$

Proceeding by induction on the dimension of the Euclidean space containing Ω, Theorem 1.3.1 follows easily from the previous argument.

<u>Corollary</u> 1.3.1. <u>In</u> $\psi(x)$ <u>is a function</u> $\gamma^{(\delta)}(\Omega)$ <u>which never vanishes then the function</u>

$$x \to 1/\psi(x)$$

belongs to $\gamma^{(\delta+1)}(\Omega)$.

Proof of Corollary 1.3.1. This is an immediate consequence of Theorem 1.3.1.

Theorem 1.3.2. Let F *be a member of* $\gamma^{(\delta_1)}(R)$. *Let* g *be a member of* $\gamma^{(\delta_2)}(\Omega;R)$. *Then* $f = F(g)$ *is a member of* $\gamma^{(\delta_1+\delta_2)}(\Omega)$.

Proof of Theorem 1.3.2. Assume first that Ω is an open subset of R^1. Then following the proof of Theorem 1.3.1 we deduce that

$$\left|\left(\frac{d}{dx}\right)^n f(x)\right| \leq$$

$$\sum_{m=1}^{n} \sum_{p=1}^{n} \sum_{i \in S(n,m,p)} \left(\frac{n!}{i!}\right) AB^m (m!)^{\delta_1} (\varepsilon D)^n \; C^m \; p^{n\delta_2 - n} \tag{1.3.11}$$

From (1.3.11) we deduce that

$$\left|\left(\frac{d}{dx}\right)^n f(x)\right| \leq$$

$$\sum_{m=1}^{n} \sum_{p=1} (p^m \; m!^{\delta_1 - 1} n!) \; A(BC)^m (\varepsilon D)^n \; p^{n\delta_2 - n} \tag{1.3.12}$$

From (1.3.12) and the hypothesis under which it holds we deduce that for every compact subset K of Ω and every $\varepsilon > 0$ there is a $C_1 > 0$ such that

$$\left|\left(\frac{d}{dx}\right)^n f(x)\right| \le (C_1\varepsilon^n) n^{(\delta_1+\delta_2)n} \tag{1.3.13}$$

for all x in K. This follows from the fact that

$$n!\ (m!)^{\delta_1 - 1} \le n!^{\delta_1}$$

and

$$p^m\ p^{n\delta_2 - n} \le n^{n\delta_2} .$$

Let $g(x)$ be a member of $\gamma^M(\Omega,R)$ where Ω is an open subset of R^1. Let $F(y)$ be a member of $\gamma^{M'}(R)$. Define $f(x) = F(g(x))$. Then by the Faa di Bruno formula we have

$$\left(\frac{d}{dx}\right)^n f(x) =$$

$$\sum_{m=1}^{n} \sum_{p=1}^{n} \sum_{i\varepsilon S(n,m,p)} \frac{n!}{i!} \left[\prod_{j=1}^{p} \left(\frac{g^{(j)}(x)}{j!}\right)^{i_j}\right] F^{(m)}(g(x)) \tag{1.3.14}$$

If x runs over a compact subset K of Ω, then $g(x)$ runs over a compact subset K' of R. Thus using the fact that $g(x)$ is in $\gamma^M(\Omega;R)$ and $F(y)$ is in $\gamma^{M'}(R)$ we deduce from (1.3.14) that for every $\varepsilon > 0$ there exist C_1 and $C_2 > 0$ such that

$$\left|\left(\frac{d}{dx}\right)^n f(x)\right| \le$$

$$\sum_{m=1}^{n} \sum_{p=1}^{n} \sum_{i\varepsilon S(n,m,p)} \left(\frac{n!}{i!}\right) \left[\prod_{j=1}^{p} \left(\frac{C_1\varepsilon^j M(j)}{j!}\right)^{i_j}\right] (C_2\varepsilon^m) M'(m) \tag{1.3.15}$$

We wish now to use a variant of Jensen's inequality for convex and concave functions to estimate the right side of (1.3.15) in certain special cases.

Lemma 1.3.2. Let us suppose that $M: R \to R_+$ is a function of x such that $[M(x)^{1/x}/x]$ is nondecreasing for $x \geq 1$, then

$$\prod_{j=1}^{p} \left(\frac{C_1 \varepsilon^j M(j)}{j!}\right)^{i_j} \leq (C_1 D\varepsilon)^n M(p)^{n/p} p^{-n} \tag{1.3.16}$$

where $1/j! \leq D^j/j^j$ for $j = 1, 2, \ldots, p$.

Proof of Lemma 1.3.2. This is a trivial application of Jensen's inequality to the measure space $\{x, \mu\}$, where $x = \{1, 2, \ldots, p\}$, and $\mu(\{j\}) = j i_j/n$.

As before we use the fact that the exponential function is convex. We observe that

$$\exp\left(\sum_{j=1}^{p} \left(\frac{i_j j}{n}\right) \ln \left[\left(\frac{C_1 \varepsilon^j M(j)}{j!}\right)^{n/j}\right]\right) \leq$$

$$\sum_{j=1}^{p} \frac{i_j i}{n} \left(\frac{C_1 \varepsilon^j M(j)}{j!}\right)^{n/j} \tag{1.3.17}$$

which is the same as saying that

$$\exp\left(\int_X f d\mu\right) \leq \int_X \exp(f)\, d\mu$$

where $f(j) = (C_1 \varepsilon^j M(j)/j!)^{n/j}$. Using Stirling's inequality it is easy to see that the right side of (1.3.17) is dominated by

$$\sum_{j=1}^{p} (i_j j/n)[(C_1 D^j \varepsilon^j M(j))/j^j]^{n/j} =$$

$$\sum_{j=1}^{p} (i_j j/n)[(C_1^{1/j}) D\varepsilon (M(j)^{1/j})/j]^n \tag{1.3.18}$$

It is in the estimation of the right side of (1.3.18) that we use the fact that $[(M(x)^{1/x})/x]$ is nondecreasing for $x \geq 1$. Using this fact we deduce that the right side of (1.3.18) and, consequently, the left side of (1.3.16) are dominated by

$$\sum_{j=1}^{p} (i_j j/n)[C_1 D\varepsilon (M(p)^{1/p})/p]^n =$$

$$(C_1 D\varepsilon)^n \; M(p)^{n/p} \; p^{-n} \tag{1.3.19}$$

which completes the proof of Lemma 1.3.2.

Again using the fact that $(M(p)^{1/p})/p$ is a nondecreasing function of p we deduce that

$$\prod_{j=1}^{p} \left(\frac{C_1 \varepsilon M(j)}{j!}\right)^{i_j} \leq (C_1 \varepsilon D)^n \; M(n) \; n^{-n} \tag{1.3.20}$$

Thus, if we suppose that $F(y)$ is a complex valued function in $\gamma^{M_2}(R)$ and $g(x)$ is a real valued function in $\gamma^{M_1}(\Omega)$, where Ω is an open subset of R, then (1.3.15) implies that if we define $f(x) = F(g(x))$, then for every $\varepsilon > 0$ and every compact subset K of Ω there exist constants C_1, C_2, D, and C_3 such that

$$\left|(d/dx)^n f(x)\right| \leq [C_3^n n^n p^m (C_1 D\varepsilon)^n M_1(n) n^{-n} (C_2 \varepsilon^m) M_2(m)]/m! \qquad (1.3.21)$$

An immediate consequence of (1.3.21) is the following.

Proposition 1.3.1. *Let* $M_2(m) = m!$ *and let* $(M_1(x)^{1/x})/x$ *be a nondecreasing function of* $x \geq 1$. *Suppose* $F(y)$ *is in* $\gamma^{M_2}(R)$ *and* $g(x)$ *is a real valued function in* $\gamma^{M_1}(\Omega)$. *Then* $f(x) = F(g(x))$ *is in* $\gamma^{\tilde{M}}(\Omega)$, *where* $\tilde{M}(n) = n^n M_1(n)$.

The more general result from which all the main results of this section follow is stated in the following Theorem.

Theorem 1.3.3. *Suppose the map*, $t \to (M_i(t)^{1/t})/t$, *is an increasing function of* $t \geq 1$. *Then* $g(x)$ *is in* $\gamma^{M_1}(\Omega,R)$ *and* $F(y)$ *is in* $\gamma^{M_2}(R)$ *implies* $f(x) = F(g(x))$ *is in* $\gamma^{M_1 M_2}(\Omega)$.

Proof. This is an immediate consequence of (1.3.21).

§2. GENERALIZED FUNCTION SPACES OF GEVREY TYPE

§2.1. Characterization of the Functions in $\gamma_c^M(R^n)$ using Paley-Wiener Theorems.

We prove in this section a theorem that is analogous to, but more general then, that given by Lemma 5.7.2 of Hörmander [13]. As before we suppose M is a mapping from R_+^n, the set of n-tuples of nonnegative numbers, into itself. We define $\gamma_c^M(R^n)$ to be the set of all ϕ in C_c^∞ such that for every $\varepsilon > 0$ there is a $C > 0$ such that

$$\left|D^\alpha\phi(x)\right| \varepsilon^{-|\alpha|}\left[\prod_{k=1}^{n} M_k(\alpha)\right]^{-1} \leq C \tag{2.1.1}$$

for all x in R^n and all α in R_+^n where $M(\alpha) = (M_1(\alpha), \ldots, M_n(\alpha))$, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$, and $D_j = -i(\partial/\partial x_j)$. We suppose always that the functions $\xi \to T_k(\xi)$ are increasing functions vanishing at zero such that

$$\left[\prod_{k=1}^{n} M_k(\alpha)\right] \leq \prod_{k=1}^{n}\left[T_k(\alpha_k)^{\alpha_k}\right] \tag{2.1.2}$$

for all n-tuples of positive numbers.

If we set

$$T(\alpha) = (T_1(\alpha_1)^{\alpha_1}, \ldots, T_n(\alpha_n)^{\alpha_n}) \tag{2.1.3}$$

then (2.1.2) implies that

$$\gamma_c^M(R^n) \subset \gamma_c^T(R^n) \,. \tag{2.1.4}$$

We define $D^\alpha\phi(\zeta)$ by the rule,

$$\zeta^\alpha\hat{\phi}(\zeta) = \int_{R^n} \exp(-i\langle x,\zeta\rangle)\ (D^\alpha\phi(x))dx\ , \tag{2.1.5}$$

which is derived by a simple integration by parts when $\alpha \in N^n$ and is taken as a definition when the coordinates of α are not integers. We are now in a position to state and prove the following generalization of the first half of Lemma 5.7.2 of Hörmander [13].

Theorem 2.1.1. Let $\xi \to G_j(\xi)$ be decreasing function of ξ in $[0,\infty]$ for $j = 1, 2, \ldots, n$. Let $\phi(x)$ be a member of $\gamma_c^M(R^n)$, where

$$M(\alpha) \leq \prod_{k=1}^{n} \left[T_k(\alpha_k)^{\alpha_k}\right] , \tag{2.1.6}$$

which vanishes outside the sphere, $\{x \in R^n: |x| \leq A\}$. Then there is for every $\varepsilon > 0$ a $K_\varepsilon > 0$ such that

$$\hat{\phi}(\xi) \leq K_\varepsilon \exp(A|\mathrm{Im}\zeta|) \prod_{j=1}^{n} \psi_j(|\mathrm{Re}\zeta|/e) \tag{2.1.7}$$

where

$$\psi_j(\xi) = G_j(\xi)^{T_j^{-1}(\xi G_j(\xi))} \tag{2.1.8}$$

Proof of Theorem 2.1.1. By the definition of the space $\gamma_c^M(R^n)$ and the relation (2.1.5) there is for every $\varepsilon > 0$ a $C_1 > 0$ depending on ε and ϕ such that for every n-tuple of positive numbers we have

$$|D^\alpha\phi(x)|\varepsilon^{-|\alpha|} \left[\prod_{k=1}^{n} T_k(\alpha_k)^{-\alpha_k}\right] \leq C_1 \tag{2.1.9}$$

From elementary properties of the Fourier integral, the Fourier transform, $\hat{\phi}(\zeta)$ of $\phi(x)$ satisfies

$$\left|\zeta^\alpha\hat{\phi}(\zeta)\right| = \left|\int_{R^n} \exp(-i\langle x,\zeta\rangle)\ (D^\alpha\phi(x))dx\right|$$

$$\leq \exp(A|\mathrm{Im}\zeta|) \int_{|x|\leq A} |D^\alpha\phi(x)|dx \tag{2.1.10}$$

In view of (2.1.9) we deduce from (2.1.10) that

$$|\hat{\phi}(\zeta)| \leq \exp(A|\mathrm{Im}\zeta|)C_1A^n \prod_{\substack{k=1\\ k\,\varepsilon\, I(k)}}^{n} \left(\frac{\varepsilon^{\alpha_k}\ T_k(\alpha_k)^{\alpha_k}}{|\mathrm{Re}\zeta_k|^{\alpha_k}}\right) \tag{2.1.11}$$

for all n-tuples of positive numbers α, where $I(k)$ is the set of k in $\{1, 2, \ldots, n\}$ such that $|\mathrm{Re}\zeta_k| \neq 0$. In case $I(k)$ is empty, we may $\alpha = (0, \ldots, 0)$ and deduce that

$$|\hat{\phi}(\zeta)| \leq \exp(A|\mathrm{Im}\zeta|) \int_{|x|\leq A} |\phi(x)|dx \tag{2.1.12}$$

Let α_k be the largest positive number such that

$$\left(\frac{\varepsilon\ T_k(\alpha_k)}{|\mathrm{Re}\zeta_k|}\right) \leq G_k(|\mathrm{Re}\zeta_k|/\varepsilon) \tag{2.1.13}$$

so that since T is an increasing function we have that

$$\alpha_k = T_k^{-1}((|\mathrm{Re}\zeta_k|/\varepsilon)\ G_k(|\mathrm{Re}\zeta_k|/\varepsilon)) \tag{2.1.14}$$

Thus, we deduce combining (2.1.13) and (2.1.14) and the definition (2.1.9) of $\psi_k(\xi)$ that

$$\left(\frac{\varepsilon\, T_k(\alpha_k)}{|\mathrm{Re}\,\zeta_k|}\right)^{\alpha_k} \leq \psi_k(|\mathrm{Re}\,\zeta_k|/\varepsilon) \tag{2.1.15}$$

from which the relation (2.1.18) follows immediately with $K_\varepsilon = C_1A^n$. This completes the proof of Theorem 2.1.1.

In developing Paley-Wiener Theorems, there are two problems to solve which can be stated as follows.

Problem 2.1.1. Given increasing functions, T_k, find decreasing functions ψ_k for which there exist positive constants C_1 and B_1 such that for all positive numbers α

$$\int_0^\infty \xi^\alpha \psi_k(\xi)d\xi \leq C_1 B_1{}^\alpha T(\alpha)^\alpha. \tag{2.1.16}$$

The solution of problem 2.1.1 for a class of functions T_k is given by the following theorem.

Theorem 2.1.2. Let T_k satisfy the condition

$$T_k(x) \leq \exp(g(x)) = S_k(x) \tag{2.1.17}$$

for all $x > 0$, where $g: R \to R$ is an increasing function which grows sufficiently slowly that the integrals

$$\int_{\xi=B}^{\infty} \xi^\alpha(\xi^{-(1/2)g^{-1}(\ln(\sqrt{\xi}))})d\xi,$$

are finite for all $\alpha > 0$. Thus, if

$$G_k(\xi) = (1/\sqrt{\xi})$$

for all $\xi > B$ and $G_k(\xi) = (1/\sqrt{B})$ for $0 \leq \xi \leq B$, we deduce that if

$$\psi_k(\xi) = G_k(\xi)^{T_k^{-1}(\xi G_k(\xi))}, \qquad (2.1.18)$$

then the integrals,

$$\int_0^\infty \xi^\alpha \psi_k(\xi) d\xi,$$

are all finite.

Proof of Theorem 2.1.2. If $G_k(\xi) \leq 1$, we observe that

$$G_k(\xi)^{T_k^{-1}(\xi G_k(\xi))} \leq G_k(\xi)^{S_k^{-1}(\xi G_k(\xi))} \qquad (2.1.19)$$

If $\xi G_k(\xi) > 0$ and $\xi > B$, then $\ell n(\xi G_k(\xi)) = \ell n(\sqrt{\xi})$. Choose B so that $\ell n(\sqrt{B}) > 0$. Then we have

$$\psi_k(\xi) = G_k(\xi)^{T_k^{-1}(\xi G_k(\xi))} \leq \exp[-(1/2)\ell n(\xi) g^{-1}((1/2)\ell n(\xi))] \qquad (2.1.20)$$

for all $\xi \geq B$. Thus,

$$\int_0^\infty \xi^\alpha \psi_k(\xi)d\xi \le \int_0^B \xi^\alpha \psi_k(\xi)d\xi +$$

$$\int_B^\infty \xi^\alpha \exp[-(1/2)\ln(\xi)g^{-1}((1/2)\ln(\xi))]d\xi \qquad (2.1.21)$$

and both integrals on the right side of (2.1.21) are finite.

Theorem 2.1.3. *Let us suppose that there are positive constants* C_1 *and* B_1 *such that*

$$\int_0^\infty \xi^\alpha \psi_k(\xi)d\xi \le C_1 B_1{}^\alpha \, T_k(\alpha)^\alpha$$

for all positive numbers α . *Then if for every* $\varepsilon > 0$ *there is a* $K_\varepsilon > 0$ *such that*

$$|\hat{\phi}(\zeta)| \le K_\varepsilon \exp(A|\mathrm{Im}\zeta|) \left[\prod_{k=1}^n \psi_k(|\mathrm{Re}\zeta_k|/\varepsilon)\right] , \qquad (2.1.22)$$

where $\hat{\phi}(\zeta)$ *is an entire function, it follows that the Fourier transform* $\phi(x)$ *of* $\hat{\phi}(\zeta)$ *given by*

$$\phi(x) = (1/2\pi)^n \int_{R^n} \exp(i\langle x,\xi\rangle)\, \hat{\phi}(\xi)d\xi \qquad (2.1.23)$$

vanishes outside the ball, $\{x: |x| \le A\}$, *and is in the space* $\gamma_c^T(R^n)$, *where* T *is given by* (2.1.3).

Proof of Theorem 2.1.3. Since $\phi(\zeta)$ is an entire function we may shift the integration into the complex domain obtaining the integral representation,

$$\phi(x) = (1/2\pi)^n \int_{R^n} \exp(i\langle x,\xi + i\eta\rangle)\, \hat{\phi}(\xi + i\eta)d\xi \ , \tag{2.1.24}$$

of $\phi(x)$ which is equivalent to that given by (2.1.23). Using (2.1.22) we see that

$$|\phi(x)| \leq K_\varepsilon (1/2\pi)^n \exp((A|\eta| - \langle x,\eta\rangle) \int_{R^n} \left[\prod_{k=1}^{n} \psi_k(|\xi|/\varepsilon)\right] d\xi . \tag{2.1.25}$$

Thus, if $|x| > A$, we deduce that if we took $\eta = (tx_1, \ldots, tx_n)/|x|$, then

$$A|\eta| - \langle x,\eta\rangle = t(A - |x|) \tag{2.1.26}$$

Substituting (2.1.26) into (2.1.25) and letting $t \to \infty$ we deduce that $|\phi(x)| \leq 0$ if $|x| > A$.

To estimate the growth of derivatives of $\phi(x)$ we use the relation,

$$D^\alpha\phi(x) = (1/2\pi)^n \int \exp(i\langle x,\xi\rangle)\xi^\alpha\hat{\phi}(\xi)d\xi \ , \tag{2.1.27}$$

and deduce that for all $\varepsilon > 0$ there is a $K_\varepsilon > 0$ such that

$$|D^\alpha\phi(x)| \leq K_\varepsilon(1/2\pi)^n \int_{R^n} \left[\prod_{k=1}^{n} |\xi_k|^{\alpha_k}\psi_k(|\xi_k|/\varepsilon)\right] d\xi \ . \tag{2.1.28}$$

for all x in R^n. Using polar coordinates in (2.1.28) we deduce that

$$\left|D^{\alpha}\phi(x)\right| \leq K_{\varepsilon} \prod_{k=1}^{n} \left(\int_0^{\infty} r^{\alpha_k} \psi_k(r/\varepsilon)dr\right) \tag{2.1.29}$$

Letting $r/\varepsilon = \xi$ we deduce that $\|\alpha\| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ implies that

$$\left|D^{\alpha}\phi(x)\right| \leq K_{\varepsilon}\varepsilon^{n}\varepsilon^{\|\alpha\|} \prod_{k=1}^{n} \left(\int_0^{\infty} \xi^{\alpha_k}\psi_k(\xi)d\xi\right) \tag{2.1.30}$$

Using the estimate in the hypothesis of Theorem 2.1.3 we deduce that

$$\left|D^{\alpha}\phi(x)\right| \leq K_{\varepsilon}\varepsilon^{n}\varepsilon^{\|\alpha\|}C_1^n B_1^{\|\alpha\|}\left[\prod_{k=1}^{n} T_k(\alpha_k)^{\alpha_k}\right] \tag{2.1.31}$$

Replacing ε by (ε/B_1) everywhere in (2.1.31) we deduce that

$$\left|D^{\alpha}\phi(x)\right| \leq K_{\varepsilon/B_1}(\varepsilon/B_1)^{n}C_1^n\varepsilon^{\|\alpha\|}\left[\prod_{k=1}^{n} T_k(\alpha_k)^{\alpha_k}\right] \tag{2.1.32}$$

Taking $\tilde{K}_{\varepsilon} = K_{\varepsilon/B_1}(\varepsilon/B_1)^{n}C_1^n$ we deduce that for every $\varepsilon > 0$ there is a $\tilde{K}_{\varepsilon} > 0$ such that

$$\left|D^{\alpha}\phi(x)\right| \leq \tilde{K}_{\varepsilon}\varepsilon^{\|\alpha\|}\left[\prod_{k=1}^{n} T_k(\alpha_k)^{\alpha_k}\right] \tag{2.1.33}$$

which implies that $\phi(x)$ is in $\gamma_C^M(R^n)$.

Remark. In order to be certain that differentiation is a continuous linear transformation of $\gamma_c^M(R^n)$ into itself it is necessary to restrict the growth of the functions T_k. A more general class of functions are those belonging to the spaces $\gamma^{M,E}(\Omega)$ where both M and E are mappings from R_+^n into R_+^n and we say that f is in $\gamma_c^{M,E}(\Omega)$ if and only if there is for every $\varepsilon > 0$ a $C > 0$ such that

$$|D^\alpha f(x)| \ \varepsilon^{-|E(\alpha)|} \left[\prod_{k=1}^{n} M_k(\alpha)\right]^{-1} \leq C$$

for all x in Ω. Then if there are constants C_1 and C_2 such that

$$M_k(\alpha) \leq C_1 \alpha^{E_k(\alpha)} C_2^{E_k(\alpha)}$$

it follows that differential operators with constant coefficients are continuous linear transformations of $\gamma^{M,E}$ into itself.

§2.2 Verification that a Function Belongs to $\gamma^{(\delta,\eta)}(\Omega)$.

If δ and η are n-tuples of positive numbers, we define $\gamma^{(\delta,\eta)}(\Omega)$, for every open subset Ω of R^n to be the set of all f in $C^\infty(\Omega)$ such that for every compact subset K of Ω and every $\varepsilon > 0$ there is a $C > 0$ such that $\|\alpha\|$

$$\sup\left\{\left|D^\alpha f(x)\right|\varepsilon^{-\|\alpha\|}\left[\prod_{k=1}^{n} \alpha_k^{\delta_k(\alpha_k)^{\eta_k}}\right] : x \in K,\quad \alpha \in N^n\right\} \leq C$$

We develop in this section techniques for verifying that a function belongs to this space.

By Proposition 1.1 of Cohoon [4] the space $\gamma^{(\delta)}(\Omega)$ is a Frechet space. Thus, the limit of a sequence of functions which is Cauchy with respect to the Frechet space topology is a member of $\gamma^{(\delta)}(\Omega)$. Also, by proposition 1.3 of Cohoon [4], the space $\gamma_c^{(\delta)}(R^n)$ is an ideal in $C_c^\infty(R^n)$ since a function in $\gamma_c^{(\delta)}(R^n)$ is in $C_c^\infty(R^n)$.

We are interested in a special class of sequences of functions in $\gamma^{(\delta)}(\Omega)$. Let $\{b_k\colon\ k = 1, 2, \ldots.\}$ be a sequence of positive numbers converging monotonically to zero. For every positive integer k, let $\psi_k(x)$ be a function in $C^\infty(R^n)$ such that $\psi_k(x) = 0$ unless $b_{k+1} < x_n < b_{k-1}$. Let K be a compact subset of R^n and let

$$\pi_k(K) = \{x \in K\colon\ b_{k+1} \leq x \leq b_{k-1}\} \qquad (2.2.1)$$

We want to determine the space $\gamma^{(\delta,n)}(R^n)$ to which the sum

$$\psi(x) = \sum_{k=1}^{\infty} \psi_k(x) \tag{2.2.2}$$

belongs.

Theorem 2.2.1. *Let* $\{\psi_k(x): k = 1, 2, \ldots\}$ *be a sequence of functions such that* (i) $\psi_k(x) \in \gamma^{(\delta)}(R^n)$, (ii) $\psi_k(x) = 0$ *in case* x *does not belong to* $[b_{k+1}, b_{k-1}]$, *where* $\{b_k: k = 1, 2, \ldots\}$ *is a sequence of positive numbers converging strictly monotonically to zero, and* (iii) *for every compact subset* K *of* R^n *and every* $\varepsilon > 0$ *there exists a* $C(K,\varepsilon) > 0$ *such that for all positive integers* k *and all n-tuples of nonnegative integers* α *we have*

$$\left|D^{\alpha}\psi_k(x)\right| \leq C(K,\varepsilon)\varepsilon^{\|\alpha\|}\|\alpha\|^{\delta\|\alpha\|}\phi(k)^{\|\alpha\|}\exp(-Ak^m) \tag{2.2.3}$$

for all x *in* K, *where* A *and* m *are positive. Let* $\psi(x)$ *be given by* (2.2.2).

(a) *Then* $\phi(x)$ *is bounded implies* $\psi(x)$ *is in* $\gamma^{(\delta)}(R^n)$.

(b) *If there exists a* $C_1 > 0$ *and a positive constant* p *such that*

$$|\phi(k)| \leq C_1 k^p \tag{2.2.4}$$

for all positive integers k, *then* $\psi(x)$ *is in* $\gamma^{(\delta')}(R^n)$ *whenever* $\delta' \geq \delta + p/m$.

(c) *If there exists a* $C_1 > 0$ *and a positive constant* p *such that*

$$|\phi(k)| \leq C_1 e^{kB} \tag{2.2.5}$$

then $\psi(x)$ *is in* $\gamma^{(\delta,\eta)}(R^n)$ *for all* $\eta > 1 + 1/(m-1)$.

Proof of (a). If $\phi(x)$ is uniformly bounded, then (2.2.3) implies

$$|D^\alpha \psi_k(x)| \leq C(K,\epsilon/B)\epsilon^{\|\alpha\|}\|\alpha\|^{\delta\|\alpha\|} \tag{2.2.6}$$

for all x in K , where B is an upper bound for the function $\phi(x)$.

Proof of (b). Suppose $\phi(x)$ satisfies (2.2.4). Then we would like to maximize

$$f(k) = k^{p\|\alpha\|}e^{-Ak^m} \tag{2.2.7}$$

Differentiating we find that $f'(k) = 0$ implies

$$k = \left(\frac{p\|\alpha\|}{Am}\right)^{1/m} \tag{2.2.8}$$

Substituting (2.2.8) into (2.2.7) we deduce that

$$f(k) \leq [(p/(Ame)^{p/m}]^{\|\alpha\|} \, \|\alpha\|^{(p/m)\|\alpha\|}$$

Thus, if $\epsilon_1 = \epsilon/[(p/(Ame)^{p/m}]$ then 2.2.3) implies

$$|D^\alpha \psi_k(x)| \leq C_1 C(K,\epsilon_1)\epsilon|\alpha||\alpha|^{(\delta+p/m)|\alpha|} \tag{2.2.9}$$

Thus, $\psi(x)$ is easily seen to be in $\gamma^{(\delta+p/m)}(R^n)$.

Proof of (c). Suppose $\phi(x)$ satisfies (2.2.5) . Then we would like to maximize

$$g(k) = e^{\|\alpha\| kB} e^{-Ak^m}. \tag{2.2.10}$$

Differentiating we find that $g'(k) = 0$ implies

$$k = \left(\frac{B\|\alpha\|}{mA}\right)^{1/(m-1)}, \tag{2.2.11}$$

where we tacitly assume $m > 1$. Substituting (2.2.11) into (2.2.10) we deduce that

$$g(k) \le C_1 \exp \left[(B/(mA))^{1/(m-1)}B - A(B/(mA))^{m/(m-1)}\right]\|\alpha\|^{m/(m-1)}. \tag{2.2.12}$$

Now we must determine whether or not the coefficient of $\|\alpha\|^{m/(m-1)}$ in the argument of the exponential function appearing in (2.2.12) is positive or negative. We use the fact that the log function is increasing to deduce that

$$(B(mA))^{1/(m-1)}B > A(B/(mA))^{m/(m-1)} \tag{2.2.13}$$

if and only if

$$(1/(m-1))[\log(B) - \log(mA)] + \log(B) >$$

$$\log(A) + (m/(m-1))[\log(B) - \log(mA)].$$

Using the fact that $m/(m-1) = 1 + 1/(m-1)$ we deduce that (2.2.13) holds if and only if

$$0 > \log(A) - \log(mA), \tag{2.2.14}$$

which is valid if and only if $0 > \log(1/m)$ which is always true provided that $m > 1$. Thus, we set

$$C_2 = (B/(mA))^{1/(m-1)}B - A(B/mA))^{m/(m-1)} \tag{2.2.15}$$

and observe that (2.2.12) and $m > 1$ imply

$$g(k) \le C_1 \exp(C_2 \|\alpha\|^{1 + 1/(m-1)}) \tag{2.2.16}$$

where $C_2 > 0$. Substituting (2.2.5), (2.2.10), and (2.2.16) into (2.2.3) we deduce that

$$\left|D^\alpha \psi_k(x)\right| \le C(K,\varepsilon)\varepsilon^{\|\alpha\|}\|\alpha\|^{\delta\|\alpha\|}C_1 \exp(C_2\|\alpha\|^{1 + 1/(m-1)}) \tag{2.2.17}$$

But if $\eta \geq 1 + 1/(m-1)$ there is for every $\varepsilon > 0$ a $\tilde{C}(K,\varepsilon) > 0$ such that the right side of (2.2.17) is dominated by

$$\tilde{C}(K,\varepsilon)\varepsilon^{\|\alpha\|}\|\alpha\|^{\delta\|\alpha\|^{\eta}} \tag{2.2.18}$$

Thus, the fact that for all $\varepsilon > 0$, there is a $\tilde{C}(K,\varepsilon) > 0$ so that

$$\left|D^\alpha\psi_k(x)\right| \leq C(K,\varepsilon)\varepsilon^{\|\alpha\|}\|\alpha\|^{\delta\|\alpha\|^n} \tag{2.2.19}$$

implies $\psi(x)$ is in $\gamma^{(\delta,n)}(R^n)$.

Proposition 2.2.1. If $\psi(x)$ is a function in $C^\infty(R^n)$ such that $\psi(x) = 0$ for $x_n \leq 0$ and $\phi(x)$ is a function in $\gamma_c^{(\delta)}(R^n)$ such that $\phi(x) = 0$ for $x_n \leq 0$, then the convolution $\phi * \psi$ belongs to $\gamma^{(\delta)}(R^n)$ and vanishes identically for $x_n \leq 0$.

Theorem 2.2.2. Let $u_k(x,t)$ be a sequence of functions in $\gamma^{(\delta)}(R_x \times \Omega)$. Let U and V be bounded open sets containing the origin of R with $\bar{U}$ contained in V. Suppose θ is a function in $\gamma^{(\delta)}(R)$ such that $\theta(x) = 1$ for $t \in R-V$ and $\theta(x) = 0$ for $t \in U$. Let $P_k(x)$ be a sequence of first degree polynomials such that P_k vanishes at b_k and such that $P_{k+1}(b_k)$ and $P_k(b_{k+1})$ are outside of $\bar{V}$, where $\{b_k: k = 1, 2, \ldots\}$ is a sequence of points of R such that $b_{k+1} < b_k$ for $k = 1, 2, \ldots,$ and $\lim_{k\to\infty} b_k = b$. Then the function $u(x,t)$ defined by

$$u(x,t) = \begin{cases} u_1(x,t)\ , & x > b_1 \\ u_k(x,t)\theta(P_{k+1}(x)) + u_{k+1}(x,t)\theta(P_k(x)),\ b_{k+1} < x \leq b_k \\ 0\ , & x \leq b \end{cases} \tag{2.2.20}$$

is a member of $\gamma^{(\delta)}(R_x \times \Omega)$ provided that $K_k = [b_{k+2}\ ,\ b_k] \times K$ implies

$$\sup\{\|\psi_k(x,t)\|_{(K_k,\varepsilon)}: k = 1, 2, \ldots\} < \infty \tag{2.2.21}$$

where

$$\psi_k(x,t) = \begin{cases} u_{k+1}(x,t)\theta(P_k(x)) \quad \text{for} \quad x \in [b_{k+1}, b_k] , \\ u_{k+1}(x,t)\theta(P_{k+2}(x)) \quad \text{for} \quad x \in [b_{k+2}, b_{k+1}] , \\ \text{and} \\ 0 \quad \text{for} \quad x \in [b_{k+2}, b_k] \end{cases} \tag{2.2.22}$$

Proof. If we define $\psi_k(x,t)$ by (2.2.22), then

$$u(x,t) = \sum_{k=1}^{\infty} \psi_k(x,t) \tag{2.2.23}$$

Let $\tilde{K}$ be an arbitrary compact subset of $R_x \times \Omega$. Then there is a $B > 0$ such that $x > B$ implies $(x,t) \in \tilde{K}$. We assume $B > b_1$. Let $\tilde{K}_k = \{(x,t) \in \tilde{K}: x \in [b_{k+2}, b_k]\}$. Then

$$\|\psi_k\|(\tilde{K}_k,\varepsilon) = \|\psi_k\|(\tilde{K},\varepsilon)$$

and Theorem 2.2.1 imply that

$$\|u\|_{(\tilde{K},\varepsilon)} \leq 2 \sup \{\|\psi_k\|_{(\tilde{K},\varepsilon)}: k = 1, 2, 3, \ldots\} \tag{2.2.24}$$

which implies that u is in $\gamma^{(\delta)}(R_x \times \Omega)$.

§2.3. Properties of the Space $\gamma^M(\Omega)$.

We define R^n_+ to be the positive cone of R^n consisting of the set of all $x = (x_1, \ldots, x_n) \in R^n$ such that $x_i > 0$ for $i=1, \ldots, n$. Let $M: N^n \to R^n_+$ be an arbitrary map. Let Ω be an open set in R^n. Define $\gamma^M(\Omega)$ to be the set of all f in $C^\infty(\Omega)$ such that for every compact subset K of Ω and every $\varepsilon > 0$ the seminorms $\|f\|^M_{(K,\varepsilon)}$ defined by

$$\|f\|^M_{(K,\varepsilon)} =$$

$$\sup\left\{\left|D^\alpha f(x)\right|\varepsilon^{-\|\alpha\|} \prod_{k=1}^{n} (M_k(\alpha)^{-1}) \quad : x \in K,\ \alpha \in N^n\right\} \tag{2.3.1}$$

are finite, where we define $\|\alpha\| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

Proposition 2.3.1. The space $\gamma^M(\Omega)$ is a Frechet space when it is equipped with the finest locally convex topology for which the seminorms (2.3.1) are continuous.

Proof of Proposition 2.3.1. It is easy to see that there is a countable basis for neighborhoods of 0 in $\gamma^M(\Omega)$. Let $\{f_m\}$ be a Cauchy sequence in $\gamma^M(\Omega)$. Then for every $r > 0$ there exists an $N(r) > 0$ such that $m_1, m_2 \geq N(r)$ imply that

$$\|f_{m_1} - f_{m_2}\|^M_{(K,\varepsilon)} < r \tag{2.3.2}$$

But (2.3.2) implies that

$$\left|D^{\alpha}f_{m_1}(x) - D^{\alpha}f_{m_2}(x)\right| \leq$$

$$r\ \varepsilon^{\|\alpha\|} \prod_{k=1}^{n} M_k(\alpha) \tag{2.3.3}$$

for all x in K and all α in N^n. Thus, (2.3.3) implies immediately that the pointwise limit f of the sequence $\{f_m\}$ is a member of $C^{\infty}(\Omega)$. Furthermore, $\{f_m\}$ converges to f in the topology of $C^{\infty}(\Omega)$.

Let $r > 0$ be given. Then the triangle inequality for the supremum seminorm implies for all positive integers m, p, and q that

$$\sup\left\{\left|D^{\alpha}f(x) - D^{\alpha}f_m(x)\right| : x \in K\right\} \varepsilon^{-\|\alpha\|}\left[\prod_{k=1}^{n} M_k(\alpha)\right]^{-1} \leq$$

$$\left[\varepsilon^{-\|\alpha\|} \prod_{k=1}^{n} M_k(\alpha)\right]^{-1} \left[\sup\left\{\left|D^{\alpha}f(x) - D^{\alpha}f_p(x)\right| : x \in K\right\} + \right.$$

$$\sup\left\{\left|D^{\alpha}f_p(x) - D^{\alpha}f_q(x)\right| : x \in K\right\} +$$

$$\left.\sup\left\{\left|D^{\alpha}f_q(x) - D^{\alpha}f_m(x)\right| : x \in K\right\}\right] \tag{2.3.4}$$

From (2.3.4) we deduce that if p, q, and m are larger than $N(r/3)$, then

$$\sup\{|D^{\alpha}f(x) - D^{\alpha}f_m(x)M: x \in K\}\varepsilon^{-\|\alpha\|} \prod_{k=1}^{n} (M_k(\alpha_k)^{-1}) \leq$$

$$\sup\{|D^{\alpha}f(x) - D^{\alpha}f_p(x)|: x \in K\}\varepsilon^{-\|\alpha\|} \prod_{k=1}^{n} (M_k(\alpha_k)^{-1}) + 2r/3 \qquad (2.3.5)$$

But there is an $N(\alpha, r/3) > 0$ such that $p > N(\alpha, r/6)$ implies the first term on the right side of (2.1.5) is smaller than $r/6$. But the left side is independent of p. Hence, $m > N(r/3)$ implies

$$\sup\{|D^{\alpha}f(x) - D^{\alpha}f_m(x)|: x \in K\}\varepsilon^{-\|\alpha\|} \prod_{k=1}^{n} (M_k(\alpha_k) < 5r/6 \qquad (2.3.6)$$

Now taking the supremum over α of the left side of (2.3.6) we deduce that $m > N(r/3)$ implies that

$$\|f - f_m\|^{M}_{(K,\varepsilon)} < r . \qquad (2.3.7)$$

Thus, $f_m \in \gamma^M(\Omega)$ and $f - f_m \in \gamma^M(\Omega)$ imply that $f \in \gamma^M(\Omega)$. Furthermore, (2.1.7) implies that the limit in $\gamma^M(\Omega)$ of $\{f_m\}$ is f. Thus, it is clear that $\gamma^M(\Omega)$ is a Frechet space.

The theorem has the following generalization.

Proposition 2.3.2. Let M be an arbitrary mapping of N^n into R^n_+. Then the space $\Gamma^M(\Omega)$, of all functions f in $C^\infty(\Omega)$ such that for every compact subset K of Ω the seminorms, $\|\ \|^M_K$, defined by

$$\|\phi\|_K^M = \sup\left\{\left|D^\alpha\phi(x)\right| \left[\prod_{k=1}^{n} (M_k(\alpha_k)^{-1})\right] : x \in K,\ \alpha \in N^n\right\} \qquad (2.3.8)$$

for ϕ in $C^\infty(\Omega)$ are finite at f , is a Frechet space.

Proof. Delete the terms $\varepsilon^{-\|\alpha\|}$ in the proof of Theorem 1.2.1. These spaces are a generalization of the spaces invented by Gevrey and studied by Roumier [19, 20], by A. Friedman, and many others.

§3. EXTENSION OF COHEN'S NONUNIQUENESS THEOREM

§3.1. Analysis of the Applicability of the Construction of Theorem 8.9.2 of Hörmander [13] in the Demonstration of Failure of the Holmgren Uniqueness Theorem When the Coefficients Are in $\gamma^{(\overline{\delta},\overline{\eta})}(R_x \times R_t)$.

In this section we determine the decay of sequences $\{b_k\}$ to zero which enable us to use the idea of the construction of Theorem 8.9.2 of Hörmander [13] to obtain a function $u(x,t)$ in $\gamma^{(\overline{\delta},\overline{\eta})}(R_x \times R_t)$ which vanishes when $x \leq 0$ together with all its derivatives and satisfies

$$[P(\partial/\partial x) - a(x,t)(\partial/\partial t)]u(x,t) = 0 \tag{3.1.1}$$

where $P(\partial/\partial x)$ is a differential operator of order $r \geq 1$ with constant coefficients, $a(x,t)$ is in $\gamma^{(\overline{\delta},\overline{\eta})}(R_x \times R_t)$, and $\overline{\delta}$ and $\overline{\eta}$ are two-tuples of positive numbers with $\eta_i \geq 1$ for $i = 1, 2$.

The main results of section 3.1 are a generalization of the techniques used in Theorem 8.9.2 and a proof of the fact that none of these generalizations will produce functions $u(x,t)$ and $a(x,t)$ in $\gamma^{(\delta)}(R_x \times R_t)$ such that $u(x,t) = 0$ for $x \leq 0$ and (3.1.1) is satisfied.

Let us set

$$\phi_k(x) = -A(k) - B(k)(x-b_k)\theta_1(\eta(k)(x-b_k)) \tag{3.1.2}$$

where $A(k)$, $B(k)$, and $\eta(k)$ are positive functions defined on the set of positive integers, $\{b_k\}$ is a sequence of positive numbers decreasing

monotonically to zero, and $\theta_1(x)$ is an increasing function of x in $\gamma^{(\delta-1)}(R)$ such that

$$\theta_1(x) = \begin{cases} -1/F & \text{when } x < -D \\ 0 & \text{when } |x| < D/2 \\ 1 & \text{when } x > D \end{cases} \tag{3.1.3}$$

Let $\theta_2(x)$ be another function in $\gamma^{(\delta-1)}(R)$ such that

$$\theta_2(x) = \begin{cases} 1 & \text{when } |x| > D \\ 0 & \text{when } |x| < D/2 \end{cases} \tag{3.1.4}$$

We assume that for every $\varepsilon > 0$ and for every compact subset K of the real line, there is a positive constant C, independent of K such that

$$\|\theta_1\|_{(K,\delta-1,\varepsilon)} \leq C$$

for all compact sets K, where for all ψ in $\gamma^{(\delta)}(R)$ we have

$$\|\psi\|_{(K,\delta,\varepsilon)} =$$

$$\sup_{x\varepsilon K} \sup_{\alpha} \left|(\partial/\partial x)^{\alpha}\psi(x)\right|\varepsilon^{-\alpha}{}_{\alpha}{}^{-\alpha\delta}$$

Let us define

$$u_k(x,t) = \exp(i\lambda_k t + \phi_k(x)) \tag{3.1.5}$$

Define

$$u(x,t) = \begin{cases} u_1(x,t) \ , \ \text{if} \ \ x \geq b_1 \\ u_k(x,t)\theta_2(n(k)(x-b_{k+1})) + u_{k+1}(x,t)\theta_2(n(k)(x-b_k)) \\ \qquad \text{if} \ \ b_{k+1} \leq x \leq b_k \ , \\ 0 \ , \ \text{if} \ \ x \leq 0 \end{cases} \tag{3.1.6}$$

We can prove that under suitable hypothesis $u(x,t)$ belongs to $\gamma^{(\delta)}(R_x \times R_t)$ and that under no additional hypothesis does

$$a(x,t) = ((\partial/\partial x)^r u(x,t))/(\partial/\partial t)u(x,t)$$

belong to $\gamma^{(\delta)}(R_x \times R_t)$.

Proposition 3.1.1. The function $u(x,t)$ defined by (3.1.6) is equal to $u_{k+1}(x,t)$ in a neighborhood of b_{k+1} provided that

$$b_{k+1} < b_k - D/n(k)$$

where D is the positive constant used in the definition of θ_2 .

Proof of Proposition 3.1.1. If $b_{k+1} < x < b_k - D/n(k)$, then $b_{k+1} - b_k < x - b_k < -D/n(k)$ implies that

$$n(k)(b_{k+1}-b_k) < n(k)(x-b_k) < -D. \tag{3.1.7}$$

But if (3.1.7) is satisfied, then

$$|n(k)(x-b_k)| > D$$

implies

$$\theta_2(\eta(k)(x-b_k)) = 1 \ ,$$

where θ_2 is given by (3.1.4). Clearly, if in addition to the supposition that (3.1.7) is satisfied we suppose that x is close enough to b_{k+1}, then we certainly make

$$|\eta(k)(x-b_{k+1})| < D/2$$

which will imply that $\theta_2(\eta(k)(x-b_{k+1})) = 0$, and that $u(x,t) = u_{k+1}(x,t)$.

Thus, from Proposition 3.1.1 we see that it is desirable to ask the sequence $\{b_k-b_{k+1}\}$ to decrease to zero in a sufficiently slow manner. We need at least

$$b_{k+1} < b_k - D/\eta(k) \tag{3.1.8}$$

Now we want to show that $u(x,t)$ is in $\gamma^{(\delta)}(R_x \times R_t)$. Note that

$$\begin{aligned}
&(\partial/\partial x)^{\alpha}(\partial/\partial t)^{\beta}\, u(x,t) = \\
&i^{\beta}\lambda_k^{\beta} \sum_{j=0}^{\beta} \binom{\beta}{j}(\partial/\partial x)^{\beta-j}u_k(x,t)\eta(k)^j\theta_2^{(j)}(\eta(k)(x-b_{k+1})) \\
&+i^{\beta}\lambda_{k+1}^{\beta} \sum_{j=0}^{\beta} \binom{\beta}{j}(\partial/\partial x)^{\beta-j}u_{k+1}(x,t)\eta(k)^j\theta_2^{(j)}(\eta(k)(x-b_k))
\end{aligned} \tag{3.1.9}$$

for $b_{k+1} \leq x \leq b_k$. Now we need to estimate the right side of (3.1.9). We use the Faa di Bruno formula to estimate $(\partial/\partial x)^{\beta-j}u_k(x,t)$. We remind ourselves that

$$u_k(x,t) = \exp(i\lambda_k t + \phi_k(x)) \ . \tag{3.1.10}$$

Hence, in view of the Faa di Bruno formula, a primary task is the computation and estimation of derivatives of $\phi_k(x)$.

We have for $n \geq 1$

$$\phi_k^{(q)}(x) = -B(k)q\ \eta(k)^{q-1}\ \theta_1^{(q-1)}(\eta(k)(x-b_k))$$

$$-B(k)\ \eta(k)^q\ \theta_1^{(q)}(\eta(k)(x-b_k))$$

Assume K is a compact subset of R. Then there exists for every $\varepsilon > 0$ a $C'_\varepsilon > 0$ independent of K such that for all x in K

$$|\phi_k^{(q)}(x)| \leq B(k)q\ \eta(k)^{q-1} C'_\varepsilon \varepsilon^{(q-1)}(q-1)^{(\delta-1)(q-1)} +$$

$$B(k)\ \eta(k)^q C'_\varepsilon \varepsilon^q\ q^{(\delta-1)q} \tag{3.1.11}$$

If $\eta(k) > 1$ for all k, then defining $(q-1)^{\delta(q-1)} = 1$ for $q = 1$ we deduce that for all positive integers q

$$q\ \eta(k)^{q-1}(q-1)^{(\delta-1)(q-1)} \leq \eta(k)^q\ q^{(\delta-1)q} \tag{3.1.12}$$

Hence, we have that for all x in K,

$$|\phi_k^{(q)}(x)| \leq (1+1/\varepsilon)C'_\varepsilon\ B(k)\ \eta(k)^q\ \varepsilon^q\ q^{(\delta-1)q}$$

Thus, there is for every $\varepsilon > 0$ a $C_\varepsilon > 0$ independent of K such that $x \in K$ implies

$$\left|\phi_k^{(q)}(x)\right| \leq C_\varepsilon\ B(k)\ \eta(k)^q\ \varepsilon^q\ q^{(\delta-1)q}. \tag{3.1.13}$$

Now we use the estimate (3.1.13) and Faa di Bruno's formula to estimate

$$(\partial/\partial x)^\beta u_k(x,t) =$$

$$\sum_{m=1}^{\alpha} \sum_{p=1}^{\alpha} \sum_{i\varepsilon S(\alpha,m,p)} \left(\frac{\alpha!}{i!}\right) \prod_{q=1}^{p} \left(\frac{\phi_j^{(q)}(x)}{q!}\right)^{i_j} u_k(x,t). \tag{3.1.14}$$

Using the fact that

$$(x-b_k)\theta_1(\eta(k)(x-b_k)) \leq 0 \tag{3.1.15}$$

for all x with just the assumption that $\eta(k) > 0$, and we have in fact assumed that $\eta(k) > 1$ for all positive integers k , it follows that

$$\left|u_k(x,t)\right| \leq \exp(-A(k)) \tag{3.1.16}$$

Combining (3.1.13), (3.1.14), and (3.1.16) we deduce that

$$\left|(\partial/\partial x)^\alpha u_k(x,t)\right| \leq$$

$$\exp(-A(k)) \sum_{m=1}^{\alpha} \sum_{p=1}^{\alpha} \sum_{i\varepsilon S(\alpha,m,p)} \left(\frac{\alpha!}{i!}\right) C_\varepsilon^m B(k)^m \eta(k)^\alpha \varepsilon^\alpha \prod_{q=1}^{p} \left(\frac{q^{(\delta-1)q}}{q!}\right)^{i_q} \tag{3.1.17}$$

Let D be a positive constant satisfying

$$\frac{1}{q!} \leq \left(\frac{D}{q}\right)^q \tag{3.1.18}$$

for all positive integers q . By Stirling's inequality

$$\sqrt{2\pi n}\ (n/e)^n < n! < \sqrt{2\pi n}\ (n/e)^n[1 + 1/(12n-1] \tag{3.1.19}$$

we see that we may take $D = e$. Thus, since $i_1 + 2i_2 + \ldots + pi_p = m$, we have

$$\prod_{q=1}^{p} \left(\frac{q^{(\delta-1)q}}{q!}\right)^{i_q} \leq e^m \prod_{q=1}^{p} (q^{(\delta-2)q})^{i_q} \tag{3.1.20}$$

Using Lemma 1.3.1 which is based upon Jensen's inequality we deduce that

$$\prod_{q=1}^{p} (q^{(\delta-2)q})^{i_q} \leq p^{\alpha\delta-2\alpha} \tag{3.1.21}$$

Combining (3.1.17), (3.1.20), and (3.1.21) we deduce that

$$\left|(\partial/\partial x)^{\alpha} u_k(x,t)\right| \leq$$

$$\exp(-A(k)) \sum_{m=1}^{\alpha} \sum_{p=1}^{\alpha} \sum_{i\varepsilon S(\alpha,m,p)} \left(\frac{\alpha!}{i!}\right)(C\varepsilon)^m B(k)^m \eta(k)^{\alpha} \varepsilon^{\alpha} p^{\alpha\delta-2\alpha} \tag{3.1.22}$$

Using the fact that

$$\sum_{i\varepsilon S(\alpha,m,p)} \frac{\alpha!}{i!} \leq (\alpha!/m!)p^m \leq \left(\frac{C_2 C_3^{\alpha} \alpha^{\alpha} p^{\alpha}}{m!}\right) \tag{3.1.23}$$

and substituting into (3.1.22) we deduce that there exist constants C_4 and C_5 so that

$$\left|(\partial/\partial x)^{\alpha}u_k(x,t)\right| \leq$$

$$\exp(-A(k))C_4C_5^{\alpha}(B(k)\eta(k))^{\alpha}\varepsilon^{\alpha}\alpha^{\alpha\delta} \tag{3.1.24}$$

Thus, we deduce that

$$\left|(\partial/\partial x)^{\alpha}(\partial/\partial t)^{\beta}u_k(x,t)\right| \leq$$

$$C_4C_5^{\alpha}(\left|\lambda_k\right|^{\beta}(B(k)\eta(k))^{\alpha}\exp(-A(k)))\varepsilon^{\alpha}\alpha^{\alpha\delta} \tag{3.1.25}$$

If $A(k)$ is chosen so that it grows sufficiently fast, then we can find for every $\varepsilon' > 0$ no matter how small, a constant C_6 so that

$$\left|\lambda_k\right|^{\beta}(B(k)\eta(k))^{\alpha}\exp(-A(k)) \leq C_6\beta^{\varepsilon'\beta}\alpha^{\varepsilon'\alpha} \tag{3.1.26}$$

for all positive integers k .

Lemma 3.1.1. The maximum of $C^{\beta}\ \beta^{-\delta'\beta}$ is given by $e^{\delta'(1/e)C^{1/\delta'}/e)}$.

Proof of Lemma 3.1.1. We can write

$$C^{\beta}\ \beta^{-\delta'\beta} = e^{\beta\ell n(C)-\delta'\beta\ell n(\beta)} \tag{3.1.27}$$

Differentiating (3.1.27) with respect to β we deduce that

$$(d/d\beta)C^{\beta}\ \beta^{-\delta'\beta} = (\ell n(C)-\delta'-\delta'\ell n(\beta))C^{\beta}\ \beta^{-\delta'\beta}\ . \tag{3.1.28}$$

The right side of (3.1.28) vanishes when $\beta = (C^{1/\delta'})/e$. Substituting back into the left side of (3.1.27) we deduce that

$$C^{\beta}\ \beta^{-\delta'\beta} \le e^{(\delta'/e)C^{1/\delta'}}$$

Lemma 3.1.2. The maximum of $C^{\beta}\ \beta^{-\delta'\beta^{\eta}}$ is given by

$$\exp\left[\frac{C^{1/(\eta\delta'\beta^{\eta-1})}}{\eta e^{1/\eta}} - \ell n(C)\left(\frac{C^{1/(\delta'\beta^{\eta-1})}}{\eta\beta^{\eta-1}e} - \frac{C^{1/(\eta\delta'\beta^{\eta-1})}}{e^{1/\eta}}\right)\right] \qquad (3.1.29)$$

where β is a solution of the transcendental equation,

$$0 = \ell n(C) - \delta'\beta^{\eta-1} - \eta\delta'\beta^{\eta-1}\ell n(\beta) \qquad (3.1.30)$$

Furthermore, the expression (3.1.29) is bounded above by AC^{θ} where $A > 0$ and $0 < \theta < 1$ provided that δ' is large enough.

Proof. There are two possibilities: $C^{1/(\beta^{\eta-1})}$ is bounded or it is not. Suppose it is bounded and the bound is equal to M. Then β must go to infinity as C goes to infinity. Then (3.1.29) is bounded above by

$$e^{(1/\eta e^{1/\eta})M^{1/\eta\delta'}}\ C^{(1/e^{1/\eta})M^{1/\eta\delta'}}$$

By taking δ' large enough we can make

$$(1/e^{1/\eta})M^{1/\eta\delta'} = \theta \qquad (3.1.31)$$

where $0 < \theta < 1$.

If $C^{(1/\beta^{\eta-1})}$ is unbounded as C goes to infinity, then we can see that the expression (3.1.29) goes to zero as C goes to infinity. In any case by

taking δ' sufficiently large we can find for all $\eta > 1$ an $A > 0$ and a real θ satisfying $e^{-1/\eta} < \theta < 1$ such that

$$C^{\beta}\ \beta^{-\delta'\beta^{\eta}} \leq AC^{\theta} \tag{3.1.32}$$

The following Lemma shows that the construction outlined in Theorem 8.9.2 of Hörmander cannot produce a solution in $\gamma^{(\delta)}(R_x \times R_t)$.

Lemma 3.1.3. There is no pair of sequences $\{A(k)\}$ and $\{\lambda_k\}$ whose absolute values diverge to plus infinity such that

$$\lim_{k\to\infty} \sup_{\alpha} \left|\lambda_k^{\alpha}\alpha^{-\delta\alpha}\exp(-A(k))\right| = 0 \tag{3.1.33}$$

and

$$\lim_{k\to\infty} \sup_{\alpha}\left(\frac{A(k)^{\alpha}(\alpha^{-\delta\alpha})}{\lambda_k}\right) = 0 \tag{3.1.34}$$

Proof of Lemma 3.1.3. According to Lemma 3.1.1 if both (3.1.33) and (3.1.34) hold it must be true that

$$\lim_{k\to\infty} \exp(\delta\ \lambda_k^{1/\delta}/e)\ \exp(-A(k)) = 0 \tag{3.1.35}$$

and

$$\lim_{k\to\infty} \left[\frac{\exp(\delta A(k)^{1/\delta}/e}{\lambda_k}\right] = 0 \tag{3.1.36}$$

Thus, we have

$$\delta A(\cdot)^{1/\delta}/e < \ell n(\lambda_k) \tag{3.1.37}$$

if k is sufficiently large. Thus, we deduce that

$$\exp(\delta\lambda_k^{1/\delta}/e)\exp(-A(k)) \geq$$

$$\exp\left[\delta\lambda_k^{1/\delta}/e - (\ell n(\lambda_k)^{e/\delta}))^{\delta}\right] \tag{3.1.38}$$

But the right side of (3.1.38) is unbounded no matter how λ_k goes to infinity since one can show that

$$\lim_{x\to\infty} \exp(\delta x^{1/\delta}/e - [\ell n(x^{e/\delta})]^{\delta}) = \infty \tag{3.1.39}$$

The following Lemma enables us, however, to produce a class of functions $\phi_k(x)$ and sequences $\{b_k : k = 1, 2, \ldots\}$ such that the construction described in the proof of Theorem 8.9.2 of Hörmander [13] will give a function $u(x,t)$ in the space $\gamma^{(\delta,\eta)}(R_x \times R_t)$ which satisfies

$$0 = (\partial/\partial x)^r u(x,t) - a(x,t)(\partial/\partial t)u(x,t)$$

everywhere in the plane, where $a(x,t)$ is also in $\gamma^{(\delta,\eta)}(R_x \times R_t)$ and $u(x,t)$ vanishes for $x \leq 0$ and yet every point of the line $x = 0$ is in the support of $u(x,t)$.

<u>Lemma</u> 3.1.4. <u>If</u> $A(k)$ <u>is asymptotic to</u> $C_1 k^s$ <u>and</u> λ_k <u>is asymptotic to</u> $C_2 \exp(C_3 k^t)$ <u>where</u> $t < s$, <u>then for every</u> $\eta > 1$ <u>there is a</u> $\delta > 0$ <u>such that</u>

$$\lim_{k\to\infty} \sup_{\beta} \left| \lambda_k^{\beta}\, \beta^{-\delta\beta^{\eta}} \exp(-A(k)) \right| = 0 \tag{3.1.40}$$

and

$$\lim_{k\to\beta} \sup_{\alpha} \left| \frac{A(k)^{\alpha} \alpha^{-\delta\alpha}}{\lambda_k} \right| = 0 \tag{3.1.41}$$

Proof of Lemma 3.1.4. First we check (3.1.40). Using Lemma 3.1.2 we observe that if δ is sufficiently large there is a $0 < \theta < 1$ such that

$$\lambda_k^{\beta} \beta^{-\delta\beta^{\eta}} \leq C_4 A C_2^{\theta} \exp(C_2^{\theta} k^{t}) \tag{3.1.42}$$

applying (3.1.42) we deduce that

$$\lambda_k^{\beta}\, \beta^{-\delta\beta^{\eta}} \exp(-A(k)) \leq C_5 \exp(C_2^{\theta} k^{t}) \exp(-C_1 k^{s})$$

from which (3.1.40) follows immediately.

According to Lemma 3.1.1 it follows that

$$\frac{A(k)^{\alpha} \alpha^{-\delta\alpha}}{\lambda_k} \leq \frac{C_5 \exp(\delta C_1^{1/\delta} k^{s/\delta}/e)}{C_2 \exp(C_3 k^{t})} \tag{3.1.43}$$

If δ is large enough, then s/δ will be smaller than t and the right side of (3.1.43) will go to zero as k goes to infinity.

§3.2. Verification of the Fact that the Holmgren Uniqueness Theorem Fails if the Coefficients Are in $\gamma^{(\overline{\delta},\overline{\eta})}(R_x \times R_t)$.

The main result of this section is the following.

Theorem 3.2.1. For every positive number s, and for every polynomial in one variable $P(X)$ of positive degree r, there are functions $a(x,t)$ and $u(x,t)$ in $\gamma^{(\overline{\delta},\overline{\eta})}(R_x \times R_t)$ such that

$$0 = P(\partial/\partial x)u(x,t) - a(x,t)(\partial/\partial t)u(x,t)$$

everywhere in $R_x \times R_t$, and $u(x,t) = 0$ for $x \leq 0$, where $\overline{\delta}$ and $\overline{\eta}$ are two-tuples of positive numbers with $\eta_1 = 1$ and $\eta_2 > s/(s-1)$ and with $\delta_i \geq 1$ for $i = 1$ and $i = 2$ and such that, furthermore, the line $x = 0$ is in the support of the function u.

Thus, we see, tactily assuming the correctness of the theorem, that we can make η_2 as close as we please to 1 by making s sufficiently large.

From the previous lemmas we know that the sequence $\{b_k : k = 1, 2, \ldots.\}$ used in the proof of the Theorem 8.9.2 cannot decay exponentially. Thus, it seems natural to assume that it decays to zero as the reciprocal of a power of k as k goes to infinity. Thus, take

$$b_k - b_{k+1} = Ek^{-s} \tag{3.2.1}$$

where E is a constant to be determined but which is larger than $2D$. Let $\phi_k(x)$ be defined by (3.1.2) but assume that

$$A(k) = k^s ,$$

$$B(k) = k^{2s} ,$$

and

$$\eta(k) = k^s \tag{3.2.2}$$

Thus, combining (3.1.45) and (3.1.2) we deduce that $\phi_k(x)$ is given by

$$\phi_k(x) = -k^s - k^{2s}(x-b_k)\theta_{(1,k)}(k^s(x-b_k)) , \tag{3.2.3}$$

where $\theta_{(1,k)}$ is a function in $\gamma^{(\delta-1)}(R)$ satisfying (3.1.3). Then (3.1.25) implies that

$$\left|(\partial/\partial x)^{\alpha}(\partial/\partial t)^{\beta}u_k(x,t)\right| \leq$$

$$C_4C_5^{\alpha}|\lambda_k|^{\beta}(k^{3s\alpha}) \exp(-k^s)\varepsilon^{\alpha}{}_{\alpha}{}^{\alpha\delta} \tag{3.2.4}$$

Now we must estimate the right side of (3.2.4). Let us find the maximum value of

$$f(x) = x^{3sa} \exp(-x^s/3) \tag{3.2.5}$$

We find that $f'(x) = 0$ implies $x = (9\alpha)^{1/s}$ so that

$$k^{3s\alpha} \exp(-k^s/3) \leq 9^{3\alpha}\ \alpha^{3\alpha} \exp(-9\alpha) \tag{3.2.6}$$

Assume that

$$\left|\lambda_k\right| \leq \lambda^k \tag{3.2.7}$$

Then we deduce that

$$\left|\lambda_k^\beta\right| \exp(-k^s/3) \leq$$

$$\exp\left[\beta\ell n(\lambda)\left(\frac{3\beta\ell n(\lambda)}{s}\right)^{1/s-1} - \left(\frac{3\beta\ell n(\lambda)}{s}\right)^{s/s-1}/3\right] \tag{3.2.8}$$

Collecting terms we see that the right side of (3.2.8) is equal to

$$\exp[\beta^{s/(s-1)}\ell n(\lambda)^{s/(s-1)}((3/s)^{1/(s-1)} - (3/s)^{s/(s-1)}/3)] \tag{3.2.9}$$

Since

$$(3/s)^{1/(s-1)} > (3/s)^{s/(s-1)}/3$$

for all $s > 1$, we see that there is a positive constant

$$C_s = \ell n(\lambda)^{s/(s-1)}((3/s)^{1/(s-1)} - (3/s)^{s/(s-1)}/3) \tag{3.2.10}$$

such that

$$\left|\lambda_k\right|^\beta \exp(-k^s/3) \leq \exp(C_s\beta^{s/(s-1)}) \tag{3.2.11}$$

<u>Lemma</u> 3.2.1. <u>If</u> $\eta \geq s/s(s-1)$, <u>then for every</u> $\varepsilon > 0$ <u>there is a</u> $C_6 > 0$ <u>such that</u>

$$|\lambda_k|^{\beta} \exp(-k^s/3) \le C_6 \beta^{\varepsilon\beta^{\eta}} \tag{3.2.12}$$

Proof of Lemma. We observe that the function

$$\exp(-\ell n(\beta)\delta\beta^{\eta} + C_5\beta^{s/(s-1)})$$

is eventually decreasing if and only if its derivative is eventually negative. But this is true only if $\eta - 1 \ge 1/(s-1)$ or equivalently only if $\eta \ge s/(s-1)$.

In view of Lemma 3.2.1 we deduce that we can make η as close to 1 as we please by making s sufficiently large. Thus, we deduce that $u_k(x,t)$ belongs to the space

$$\gamma^{((\delta,\varepsilon);(1,s/(s-1)))}(R_x \times R_t) = \gamma^{(\bar{\delta},\bar{\eta})}(R_x \times R_t) \tag{3.2.13}$$

where $\bar{\delta} = (\delta,\varepsilon)$ and $\bar{\eta} = (1,s/(s-1))$. Now we want to define

$$a(x,t) = P(\partial/\partial x)u(x,t)/(\partial/\partial t)u(x,t) , \tag{3.2.14}$$

where u(x,t) is given by (3.1.6), and prove that $a(x,t)$ belongs to $\gamma^{(\bar{\delta}',\bar{\eta}')}$ for some choice of two-tuples $\bar{\delta}'$ and $\bar{\eta}'$ of positive numbers.

Let I_k^1 , I_k^2 , and I_k^3 be subintervals of $[b_{k+1},b_k]$ defined by the rules,

$$I_k^1 = \{x:\ b_{k+1} \le x \le b_{k+1} + Dk^{-s}\}$$

$$I_k^2 = \{x:\ b_{k+1} + Dk^{-s} \le x \le b_k - Dk^{-s}\}$$

$$I_k^3 = \{x:\ b_k - Dk^{-s} \le x \le b_k\}\ . \tag{3.2.15}$$

We must investigate the growth of the derivatives of $a(x,t)$ in each of the above classes of intervals.

First suppose that x belongs to I_k^2. Then applying (3.2.15) and (3.2.2) we deduce that in this interval

$$-E + D \le \eta(k)(x-b_k) \le -D \tag{3.2.16}$$

and

$$D \le \eta(k)(x-b_{k+1}) \le E - D \tag{3.2.17}$$

From (3.2.16), (3.2.17), (3.1.4), and (3.1.6) we deduce that $x \in I_k^2$ implies

$$u(x,t) = u_k(x,t) + u_{k+1}(x,t) \tag{3.2.18}$$

and that

$$a(x,t) = \frac{P(\partial/\partial x)[u_k(x,t) + u_{k+1}(x,t)]}{i\lambda_k u_k(x,t) + i\lambda_{k+1} u_k(x,t)} \tag{3.2.19}$$

But if x is in I_k^2, then (3.1.3) and (3.1.2) imply that

$$\phi_k(x) = -k^s - k^{2s}(x-b_k)(-1/F) \tag{3.2.20}$$

and

$$\phi_{k+1}(x) = -k^{s} - k^{2s}(x-b_{k+1}) \tag{3.2.21}$$

Thus, combining (3.2.19), (3.2.20), and (3.2.21) we deduce that

$$a(x,t) = \frac{P(k^{2s}/F)u_k(x,t) + P(-(k+1)^{2s})u_{k+1}(x,t)}{i\lambda_k u_k(x,t) + i\lambda_{k+1}u_{k+1}(x,t)} \tag{3.2.22}$$

Thus, dividing numerator and denominator of (3.2.22) by $\lambda_k \lambda_{k+1}$ we deduce that

$$a(x,t) = \frac{\dfrac{P(k^{2s}/F)}{i\lambda_k}\dfrac{u_k(x,t)}{\lambda_{k+1}} + \dfrac{P(-(k+1)^{2s})}{i\lambda_{k+1}}\dfrac{u_{k+1}(x,t)}{\lambda_k}}{\dfrac{u_k(x,t)}{\lambda_k} + \dfrac{u_{k+1}(x,t)}{\lambda_{k+1}}} \tag{3.3.23}$$

If we choose λ_k so that

$$\frac{P(k^{2s}/F)}{i\lambda_k} = \frac{P(-(k+1)^{2s})}{i\lambda_{k+1}}, \tag{3.2.24}$$

then factor the common factor (3.2.24) out of the numerator of (3.2.23) we see that in I_k^2 we have

$$a(x,t) = \frac{P(k^{2s}/F)}{i\lambda_k}. \tag{3.2.25}$$

We must show that (3.2.24) implies that the right side of (3.2.25) goes to zero as k goes to infinity. From (3.2.24) we deduce that

$$\left| \frac{\lambda_k}{\lambda_{k+1}} \right| = \frac{|P(k^{2s}/F)|}{|P(-(k+1)^{2s})|} \tag{3.2.26}$$

Thus, if the degree of $P(\partial/\partial x)$ is r there is for every $C_7 > F^r$ a $C_8 > 0$ and a $k_0 > 0$ such that if $k > k_0$, then

$$|\lambda_k| \leq C_8 C_7^k \tag{3.2.27}$$

This is obvious in the case $P(\partial/\partial x) = (\partial/\partial x)^r$, since in this case we know that

$$|\lambda_k| = (k-1)^{2s}(F^r)^{k-1}\lambda_1 \tag{3.2.28}$$

is the unique solution of the difference equation for $|\lambda_k|$ defined by (3.2.26). Thus, for every $F > 1$, it is obvious that

$$\lim_{k\to\infty} \sup_{x\in I_k^2} |a(x,t)| = 0 \tag{3.2.29}$$

since an exponential decay will always suppress a polynomial growth.

Now suppose that x belongs to I_k^1. In other words, assume that x satisfies the inequality

$$b_{k+1} \leq x \leq b_{k+1} + Dk^{-s} \tag{3.2.30}$$

Then

$$-E \leq \eta(k)(x-b_k) \leq -E + D \tag{3.2.31}$$

and

$$0 \leq \eta(k+1)(x-b_{k+1}) \leq D(k+1/k)^s \tag{3.2.32}$$

Thus, $x \in I_k^1$ implies

$$\theta_1(\eta(k)(x-b_k)) = -1/F \tag{3.2.33}$$

and

$$0 \leq \theta_1(\eta(k+1)(x-b_{k+1})) \leq 1 \tag{3.2.34}$$

From (3.2.31), (3.2.2), and (3.2.33) it follows that

$$-Ek^{-s}(-1/F) \geq (x-b_k)\theta_1(\eta(k)(x-b_k)) \geq -((E-D)/F)k^s \tag{3.2.35}$$

Multiplying all terms of (3.2.35) by $-B(k)$ and reversing the inequalities again we deduce that

$$-(E/F)k^s \leq -B(k)(x-b_k)\theta_1(\eta(k)(x-b)) \leq -((E-D)/F)k^s \tag{3.2.36}$$

Adding $-A(k)$ to each term of (3.2.36) we deduce that

$$-(1+E/F)k^s \leq \phi_k(x) \leq -(1+(E-D)/F)k^s \tag{3.2.37}$$

Now we want to estimate $\phi_{k+1}(x)$ in this interval. From (3.2.34) and (3.2.2) we deduce that

$$0 \leq B(k+1)(x-b_{k+1})\theta_1(n(k+1)(x-b_{k+1})) \leq$$

$$(k+1)^{2s}(x-b_{k+1}) \leq (k+1)^{2s}k^{-s}D \tag{3.2.38}$$

Using (3.2.2) and (3.1.2) we see after multiplying all terms of (3.2.28) by -1, reversing the inequalities and adding $-A(k+1)$ to all terms of (3.2.38) that

$$-(k+1)^s \geq \phi_{k+1}(x) \geq -(k+1)^s(1+(\frac{k+1}{k})^s D) \tag{3.2.39}$$

Combining (3.2.39) and (3.2.37) we deduce that in I_k^1

$$\phi_{k+1}(x) - \phi_k(x) \geq$$

$$-(k+1)^s(1+(\frac{k+1}{k})^s D) + (1+(E-D)/F)k^s \tag{3.2.40}$$

We observe that $x \varepsilon I_k^1$ implies

$$\phi_{k+1}(x) - \phi_k(x) \geq ((E-D)/F - ((k+1)/k)^{2s}D)k^s - ((k+1)^s-k^s) \tag{3.2.41}$$

Observe that $(k+1)/k \leq 2$ for all positive integers k. This follows from the fact that $(x+1)/x$ is a decreasing function on the positive x-axis. Hence, we note that in I_k^1

$$\phi_{k+1}(x) - \phi_k(x) \geq ((E-D)/F - 4^s D)k^s - ((k+1)^s - k^s) \tag{3.2.42}$$

Note that since $2^s k^s \geq (k+1)^s - k^s$ it follows that

$$\phi_{k+1}(x) - \phi_k(x) \geq ((E-D)/F - 4^s D - 2^s)k^s \tag{3.2.43}$$

Thus, we need to make sure that

$$E > D + 4^s DF + 2^s F \tag{3.2.44}$$

Indeed it is easy to choose E so that when x is in I_k^1,

$$\phi_{k+1}(x) - \phi_k(x) \geq 4k^s \tag{3.2.45}$$

We can write

$$(\partial/\partial t)u(x,t) = i\lambda_{k+1}u_{k+1}(x,t)(1+\eta_k(x,t)) . \tag{3.2.46}$$

This follows from the fact that in I_k^1 the function $u(x,t)$, in view of (3.1.6), (3.1.3), and (3.2.2), can be expressed as

$$u(x,t) = u_{k+1}(x,t) + u_k(x,t)\theta_2(k^s(x-b_{k+1})) , \tag{3.2.47}$$

and, consequently, in (3.2.46) we may take

$$\eta_k(x,t) = \lambda_k\lambda_{k+1}^{-1}\theta_2(k^s(x-b_{k+1}))\exp[i(\lambda_k-\lambda_{k+1})t - (\phi_{k+1}(x)-\phi_k(x))]$$

Applying Liebniz's formula we deduce from the above formula that

$$\left|(\partial/\partial x)^\alpha(\partial/\partial t)^\beta\eta_k(x,t)\right| \leq$$

$$\left|G(k,\beta)\right| \sum_{j=0}^{\alpha} \binom{\alpha}{j}k^{s(\alpha-j)}\left|\theta_2^{(\alpha-j)}(k^s(x-b_{k+1}))\right|\left|\left(\frac{\partial}{\partial x}\right)^j H_k(x,t)\right| \tag{3.2.48}$$

where

$$G_{(k,\beta)} = \lambda_k \lambda_{k+1}^{-1} (i(\lambda_k - \lambda_{k+1}))^{\beta} \tag{3.2.49}$$

and

$$H_k(x,t) = \exp[i(\lambda_k - \lambda_{k+1})t - (\phi_k(x) - \phi_{k+1}(x))] \tag{3.2.50}$$

Applying the Faa di Bruno formula to (3.2.50) we deduce that

$$\left(\frac{\partial}{\partial x}\right)^j H_k(x,t) =$$

$$\sum_{m=1}^{j} \sum_{p=1}^{j} \sum_{q\varepsilon S(j,m,p)} \frac{j!}{q!} \left[\prod_{n=1}^{p} \left(\frac{\phi_k^{(n)}(x) - \phi_{k+1}^{(n)}(x)}{n!} \right)^{q_n} \right] H_k(x,t) \tag{3.2.51}$$

Using (3.1.13) and (3.2.2) we deduce that for every $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that for all x in R

$$\left|\phi_k^{(n)}(x)\right| \le C_\varepsilon k^{2s+ns} \varepsilon^n n^{(\delta-1)n}$$

and

$$\left|\phi_{k+1}^{(n)}(x)\right| \le C_\varepsilon (k+1)^{2s+ns} \varepsilon^n n^{(\delta-1)n} \tag{3.2.52}$$

Using (3.2.52) and (3.2.51) we deduce that

$$\left|\left(\frac{\partial}{\partial x}\right)^j H_k(x,t)\right| \leq$$

$$\left|H_k(x,t)\right| \sum_{m=1}^{j} \sum_{p=1}^{j} \sum_{q \epsilon S(j,m,p)} \left(\frac{j!}{q!}\right) \varepsilon^j 2^m \left[\prod_{n=1}^{p} \frac{C_\varepsilon^{q_n} (k+1)^{(2s+ns)^{q_n}} n^{(\delta-1)^{nq_n}}}{(n!)^{q_n}} \right] \tag{3.2.53}$$

Applying (3.1.20) and (3.1.21) to (3.2.53) and remembering that

$$q_1 + q_2 + \dots + q_p = m \quad \text{and} \quad 1q_1 + 2q_2 + \dots + pq_p = j$$

we observe that

$$\left|\left(\frac{\partial}{\partial x}\right)^j H_k(x,t)\right|$$

$$\left|H_k(x,t)\right| \sum_{m=1}^{j} \sum_{p=1}^{j} \sum_{q \epsilon S(j,m,p)} \left(\frac{j!}{q!}\right) \varepsilon j 2^m C_\varepsilon^m (k+1)^{2sm+sj} p^{j\delta-2j} \tag{3.2.54}$$

Using (3.1.23) we deduce that there exist positive constants C_4 and C_5 such that

$$\left|\left(\frac{\partial}{\partial x}\right)^j H_k(x,t)\right| \leq$$

$$\left|H_k(x,t)\right| \left[C_4 C_5^j \varepsilon^j (k+1)^{3sj} j^{j\delta} \right] \tag{3.2.55}$$

We note that for every $\varepsilon > 0$ *there exists a* $C_6 > 0$ *such that*

$$\left|\theta_2^{(\alpha-j)}(k^s(x-b_{k+1}))\right| \leq C_6\varepsilon^{\alpha-j}(\alpha-j)^{(\alpha-j)(\delta-1)} \tag{3.2.56}$$

Observing that $k^{s(\alpha-j)} \leq (k+1)^{3s(\alpha-j)}$ and substituting (3.2.45), (3.2.55), and (3.2.56) into (3.2.48) we observe that there exist constants C_7 and C_8 such that

$$\left|(\partial/\partial x)^{\alpha}(\partial/\partial t)^{\beta}\eta_k(x,t)\right| \leq$$

$$\left|G_{(k,\beta)}\right| C_7 C_8{}^{\alpha}\varepsilon^{\alpha}\alpha^{\alpha\delta}(k+1)^{3s\alpha}\exp(-4k^s). \tag{3.2.57}$$

Observing that

$$\left(\frac{k+1}{k}\right)^{3s\alpha} \leq 2^{3s\alpha}$$

and substituting (3.2.28) into (3.2.49) we see that

$$\left|(\partial/\partial x)^{\alpha}(\partial/\partial t)^{\beta}\eta_k(x,t)\right| \leq$$

$$\exp(-4k^s)[(1/F^r)2^{\beta}k^{2s\beta}(F^r)^{k\beta}\lambda_1^{\beta}C_7C_8^{\alpha}\varepsilon^{\alpha}\alpha^{\alpha\delta}2^{3s\alpha}k^{3s\alpha}] . \tag{3.2.58}$$

To estimate the right side of (3.2.58) we observe that

$$k^{2s\beta}\exp(-k^s) \leq (2\beta)^{2\beta}\exp(-2\beta) ,$$

$$(F^r)^{k\beta}\exp(-k^s) \leq \exp((\ln(F^r)\beta)^{s/(s-1)}-(2\beta)) ,$$

and

$$k^{3s\alpha}\exp(-k^s) \leq (3\alpha)^{3\alpha}\exp(-3\alpha) \tag{3.2.59}$$

From (3.2.59) we deduce that for every $\varepsilon > 0$ and for every $\delta_2 > 0$ there exists a constant $C_9 > 0$ such that for all x in I_k^1

$$\left|(\partial/\partial x)^{\alpha}(\partial/\partial t)^{\beta}\eta_k(x,t)\right| \varepsilon^{-\alpha-\beta}\beta^{-\delta_2\beta^{\eta_2}}\alpha^{-\delta_1\alpha} \leq$$

$$C_9 \exp(-k^s) \tag{3.2.60}$$

where $\eta_2 \geq s/(s-1)$ and $\delta_1 > \delta+3$.

Further, we write

$$\Phi_k(x) = \exp(-\phi_{k+1}(x))\; P(\partial/\partial x)\; \exp(\phi_{k+1}(x)) \tag{3.2.61}$$

and

$$\psi_k(x) = \exp(-\phi_{k+1}(x))\; P(\partial/\partial x)[\theta_2(k^s(x-b_{k+1}))\; \exp(\phi(x))] \tag{3.2.62}$$

It is clear that we can estimate the derivatives of $\Phi_k(x)$ by powers of k^s and the derivatives of $\psi_k(x)$ by powers of k^s times $\exp(-4k^s)$. Thus, writing

$$P(\partial/\partial x)u(x,t) =$$

$$u_{k+1}(x,t)[\Phi_k(x) + \exp(i(\lambda_k-\lambda_{k+1})t)\psi_k(x)] \tag{3.2.63}$$

we see that

$$a(x,t) = \frac{\Phi_k(x) + \exp(i(\lambda_k-\lambda_{k+1})t)\psi_k(x)}{i\lambda_{k+1}(1+\eta_k(x,t))} \tag{3.2.64}$$

It is easy to see that for every $\varepsilon > 0$ there is a $C_{10} > 0$ such that $x \in I_k^1$ implies

$$\left|(\partial/\partial x)^{\alpha}(\partial/\partial t)^{\beta}a(x,t)\right| \varepsilon^{-\alpha-\beta}\alpha^{-\alpha\delta_1}\beta^{-\beta^{\eta_2}\delta_2} \leq$$

$$C_{10} / \sqrt{|\lambda_{k+1}|} \tag{3.2.65}$$

if δ_1 is sufficiently large, where $\eta_2 \geq s/(s-1)$.

We show that if x is in I_k^3, then

$$\phi_k(x) - \phi_{k+1}(x) \geq 4k^s$$

if E is sufficiently large and we repeat the previous argument to obtain an estimate similar to (3.2.65). This completes the proof.

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